

2024.1.8 SODA 2024 @ Alexandria, Virginia, U.S.

Gap Amplification for Reconfiguration Problems

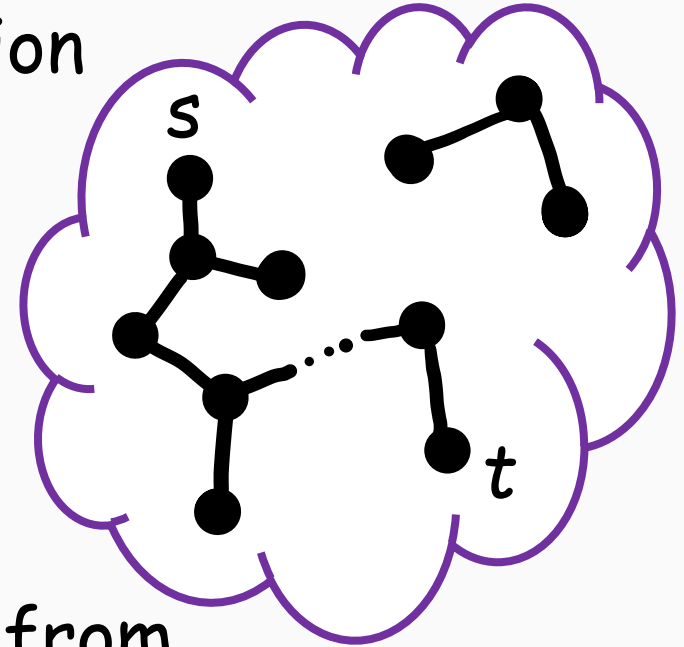
Naoto Ohsaka

(CygerAgent, Inc.)

Intro of reconfiguration

Imagine **connecting** a pair of feasible solutions (of NP problem)
under a particular adjacency relation

- Q. Is a pair of solutions reachable to each other?
- Q. If so, what is the shortest transformation?
- Q. If not, how can the feasibility be relaxed?



Many reconfiguration problems have been derived from

Satisfiability, Coloring, Vertex Cover, Clique, Dominating Set, Feedback Vertex Set, Steiner Tree, Matching, Spanning Tree, Shortest Path, Set Cover, Subset Sum, ...

See [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
[Nishimura. Algorithms 2018] [van den Heuvel. Surv. Comb. 2013]
[Hoang. <https://reconf.wikidot.com/>]

Example

3-SAT Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

- **Input:** 3-CNF formula φ & satisfying σ_s, σ_t
- **Output:** $\sigma = \langle \sigma^{(0)} = \sigma_s, \dots, \sigma^{(\ell)} = \sigma_t \rangle$ (reconf. sequence) s.t.
 - $\sigma^{(i)}$ satisfies φ (feasibility)
 - $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)

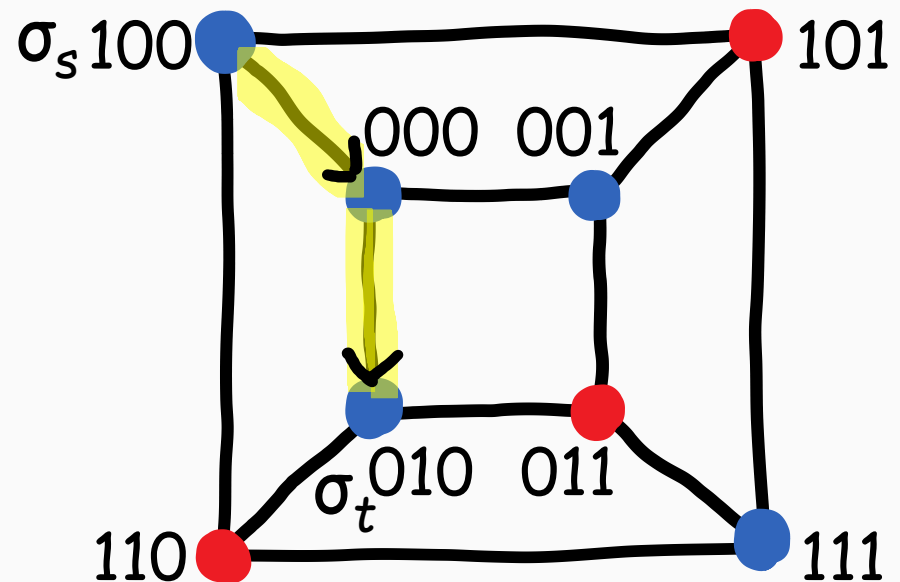
YES case

$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

$$\sigma_s = (1,0,0)$$

$$\sigma_t = (0,1,0)$$

⚠ Length of σ can be $2^{\Omega(\text{input size})}$



Example

3-SAT Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

- **Input:** 3-CNF formula φ & satisfying σ_s, σ_t
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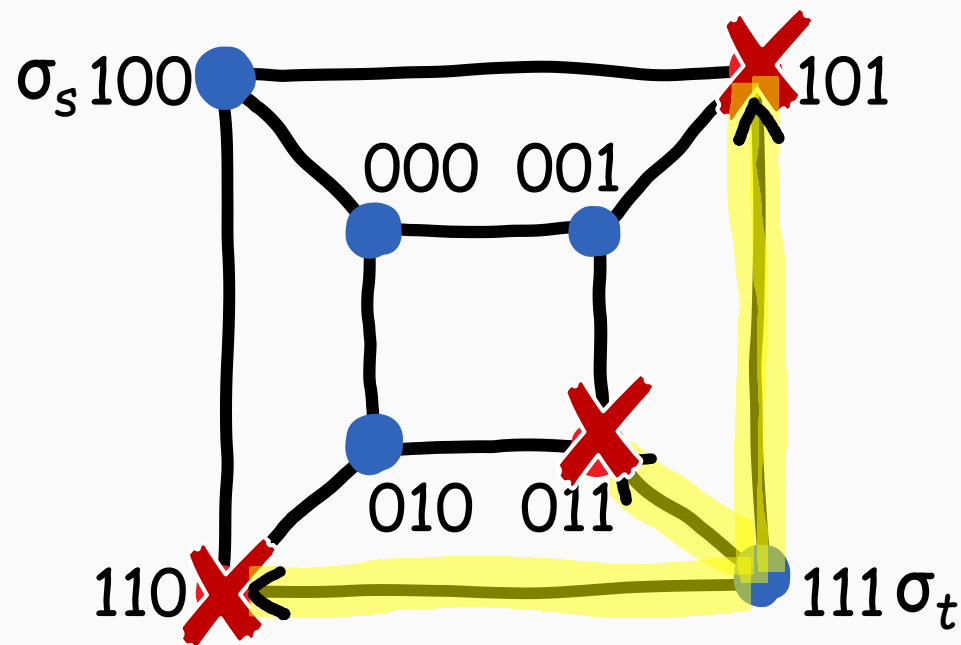
NO case

$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

$$\sigma_s = (1,0,0)$$

$$\sigma_t = (1,1,1)$$

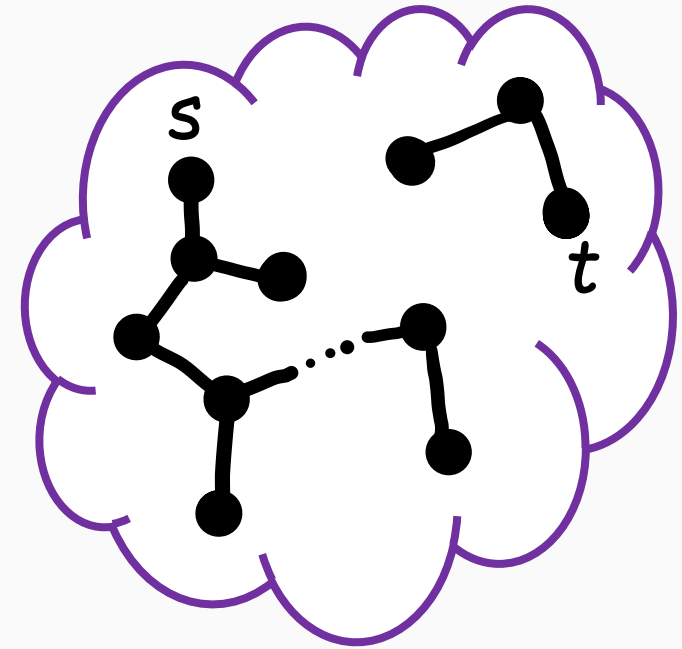
⚠ Length of σ can be $2^{\Omega(\text{input size})}$



Optimization variants of reconfiguration problems

Even if...

- 😞 **NOT** reconfigurable! and/or
- 😞 many problems are **PSPACE-complete!**



Still want an "approximate" reconf. sequence
(e.g.) made up of almost-satisfying assignments



Relax feasibility to obtain approximate reconfigurability

e.g. Set Cover Reconf.

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Subset Sum Reconf. [Ito-Demaine. J. Comb. Optim. 2014]

Submodular Reconf. [O.-Matsuoka. WSDM 2022]

Example+

Maxmin 3-SAT Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

- **Input:** 3-CNF formula φ & satisfying σ_s, σ_t
- **Output:** $\sigma = \langle \sigma^{(0)} = \sigma_s, \dots, \sigma^{(\ell)} = \sigma_t \rangle$ (reconf. sequence) s.t.
 - ~~$\sigma^{(i)}$ satisfies φ~~ (feasibility)
 - $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)
- **Goal:** $\max_{\sigma} \text{val}_{\varphi}(\sigma) \stackrel{\text{def}}{=} \min_i (\text{frac. of satisfied clauses by } \sigma^{(i)})$

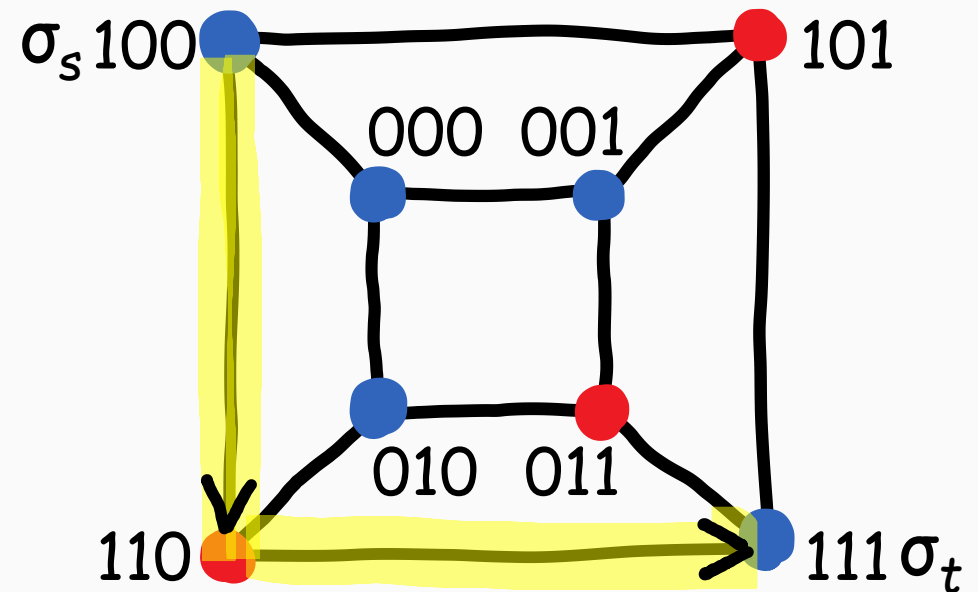
$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

- $\sigma_s = (1,0,0)$

- $\sigma_t = (1,1,1)$

→ $\text{val}_{\varphi}(\sigma) = \min \{1, \frac{2}{3}, 1\} = \frac{2}{3}$

⚠ Length of σ can be $2^{\Omega(\text{input size})}$



Known results on hardness of approximation

NP-hardness of approx. for Maxmin SAT & Clique Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

- Not optimal \because SAT & Clique Reconf. are **PSPACE**-complete
- Rely on **NP**-hardness of approximating Max SAT & Max Clique

Significance of showing **PSPACE**-hardness

- **no polynomial-time** algorithm ($P \neq PSPACE$)
- **no polynomial-length** sequence ($NP \neq PSPACE$)

(probabilistically checkable proof)



Reconfiguration analogue of the PCP theorem

[Arora-Lund-Motwani-Sudan-Szegedy. J. ACM 1998] [Arora-Safra. J. ACM 1998]

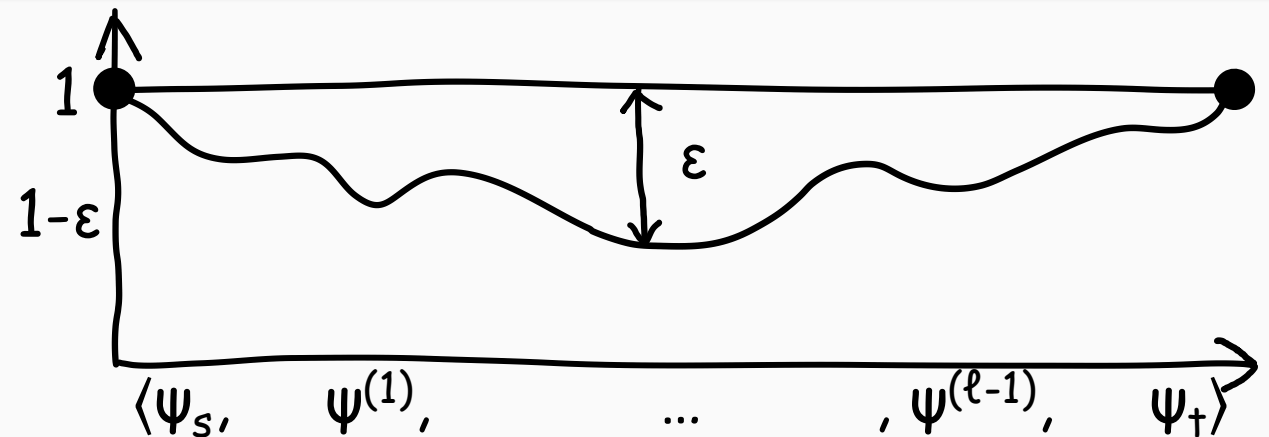
Our working hypothesis [O. STACS 2023]

Reconfiguration Inapproximability Hypothesis (RIH)

Binary CSP G & satisfying ψ_s, ψ_t , **PSPACE**-hard to distinguish btw.

- (Completeness) $\exists \Psi \text{ val}_G(\Psi) = 1$ (some sequence violates **no** constraint)
- (Soundness) $\forall \Psi \text{ val}_G(\Psi) < 1 - \varepsilon$ (any sequence violates **> ε -frac.** of constraints)

- \Rightarrow **PSPACE**-hard to approx.
Maxmin Binary CSP Reconf.
- True if "NP-hard" is used
[Ito et al. Theor. Comput. Sci. 2011]



Under **RIH**, many problems are **PSPACE**-hard to approximate via gap-preserving reductions [O. STACS 2023]

Limitation of [O. STACS 2023]

🙄 Inapprox. factors are not explicitly shown

Recall from [O. STACS 2023]

- RIH claims " $\exists \varepsilon$ Gap[1 vs. $1-\varepsilon$] Binary CSP Reconf. is PSPACE-hard"
- Can reduce to Gap[1 vs. $1-\delta$] ** Reconf.

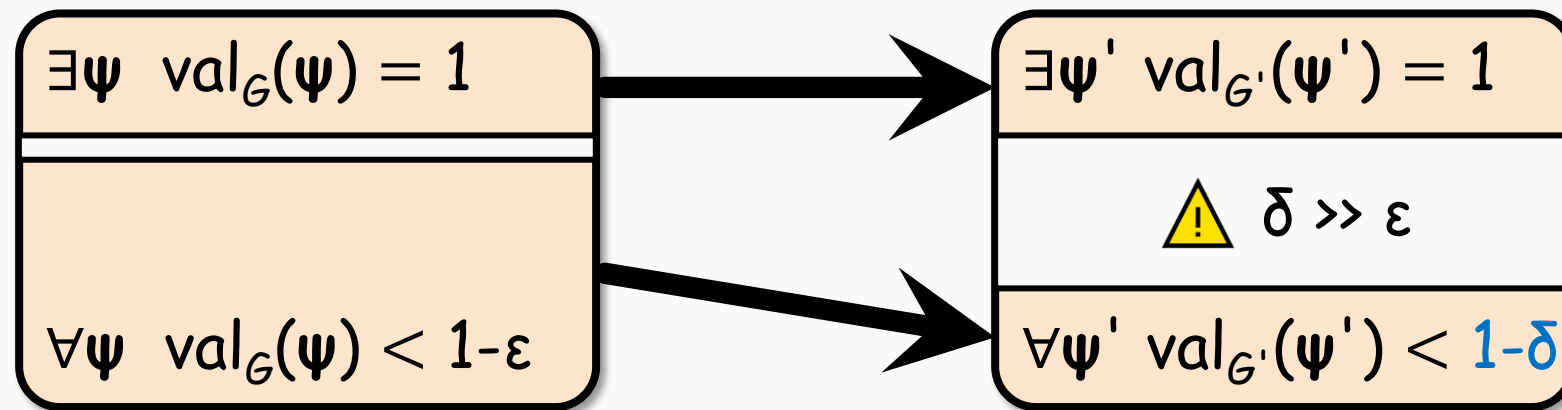
⚠️ δ (as well as ε) can be arbitrarily small, because...

- δ depends on ε (e.g., $\delta = \varepsilon^2$)
- RIH doesn't specify any value of ε
 - May not rule out $0.999\dots999$ -approx. algorithm

🎯 Wanna say Gap[1 vs. 0.999] ** Reconf. is PSPACE-hard
only assuming RIH

Our target: Gap amplification

- (Polynomial-time) reduction that makes a tiny gap into a larger gap



In NP world...

The parallel repetition theorem [Raz. SIAM J. Comput. 1998]

→ 😊 Gap[1 vs. 0.000...001] Binary CSP is NP-hard (i.e. gap ≈ 1)

In reconfiguration world...

😞 Naive parallel repetition fails to amplify gap ϵ of
Gap[1 vs. $1 - \epsilon$] Binary CSP Reconf. [O. arXiv 2023]

Our target: Gap amplification

- (Polynomial-time) reduction that makes a tiny gap into a larger gap

$\exists \psi \text{ val}_G(\psi)$

$\text{val}_{G'}(\psi')$

Can we derive explicit factors of
PSPACE-hardness of approx.
only assuming RIH?

In NP world...



(i.e. gap ≈ 1)

In reconfigur... world...

 Naive parallel repetition fails to amplify gap ε of
Gap[1 vs. $1-\varepsilon$] Binary CSP Reconf. [[O. arXiv 2023](#)]

Our results

😊 Can derive explicit inapproximability factors only assuming **RIH**!!

	Maxmin Binary CSP Reconfiguration	Minmax Set Cover Reconfiguration
PSPACE -hardness under RIH	0.9942 (this paper)	1.0029 (this paper)
NP -hardness rely on parallel repetition theorem [Raz. SIAM J. Comput. 1998]	>0.75 (this paper) 0.993 [Ito et al. Theor. Comput. Sci. 2011] [O. STACS 2023]	1.0029 (this paper)
approximability	≈0.25 [O. arXiv 2023]	2 [Ito et al. Theor. Comput. Sci. 2011]

Main result

Gap amplification for Binary CSP Reconf.

- We prove gap amplification à la Dinur [Dinur. J. ACM 2007]

(Informal) For any small const. $\varepsilon \in (0,1)$,

gap	alphabet size	degree	spectral expansion
1 vs. $1-\varepsilon$	W	d	λ
1 vs. $1-0.0058$	$W' = W d^{O(\varepsilon^{-1})}$	$d' = \left(\frac{d}{\varepsilon}\right)^{O(\varepsilon^{-1})}$	$\lambda' = O\left(\frac{\lambda}{d}\right) d'$

- 😊 Can make λ'/d' arbitrarily small by decreasing λ/d
- 😞 Alphabet size W' gets gigantic depending on ε^{-1}

Related work

Probabilistically checkable debates — PCP-like charact. of **PSPACE**

[Condon-Feigenbaum-Lund-Shor. Chic. J. Theor. Comput. Sci.'95]

- \Rightarrow Quantified Boolean Formula is **PSPACE**-hard to approx.

Other optimization variants of reconfiguration (orthogonal to this study)

- Shortest sequence finding

[Bonamy-Heinrich-Ito-Kobayashi-Mizuta-Mühlenthaler-Suzuki-Wasa. STACS 2020]

[Ito-Kakimura-Kamiyama-Kobayashi-Okamoto. SIAM J. Discret. Math. 2022]

[Kamiński-Medvedev-Milanič. Theor. Comput. Sci. 2011]

[Miltzow-Narins-Okamoto-Rote-Thomas-Uno. ESA 2016]

- Incremental optimization

[Blanché-Mizuta-Ouvrard-Suzuki. IWOCA 2020]

[Ito-Mizuta-Nishimura-Suzuki. J. Comb. Optim. 2022]

[Yanagisawa-Suzuki-Tamura-Zhou. COCOON 2021]

In the remainder of this talk...

Sketch of gap amplification

1. Preprocessing step

- Degree reduction [O. STACS 2023]
- Expanderization (skipped)

2. Powering step

- Simple appl. of [Dinur. J. ACM 2007] [Radhakrishnan.ICALP 2006] to Binary CSP Reconf. loses perfect completeness
- **TRICK: Alphabet squaring [O. STACS 2023] & modified verifier**

Recap: Max Binary CSP

Dinur's gap amplification [Dinur. J. ACM 2007]

- **Input:** Binary CSP $G = (V, E, \Sigma, \Pi = (\pi_e)_{e \in E})$
- **Output:** $\psi: V \rightarrow \Sigma$
 ψ satisfies (v, w) if $(\psi(v), \psi(w)) \in \pi_{(v, w)}$
- **Goal:** $\max_{\psi} \text{val}_G(\psi) \stackrel{\text{def}}{=} (\text{frac. of edges satisfied by } \psi)$

Example

- 3-Coloring: $\Sigma = \{R, G, B\}$, $\pi_e = \{(R, G), (G, R), (G, B), (B, G), (B, R), (R, B)\}$
- 2-SAT: $\Sigma = \{0, 1\}$, $\pi_C = \{\text{asgmt. satisfying 2-literal clause } C\}$

(Completeness) $\exists \psi \text{ val}_G(\psi) = 1 \implies \exists \psi' \text{ val}_{G'}(\psi') = 1$

(Soundness) $\forall \psi \text{ val}_G(\psi) < 1 - \epsilon \implies \forall \psi' \text{ val}_{G'}(\psi') < 1 - \Omega(T \cdot \epsilon)$

const. parameter

Recap: Dinur's gap amplification [Dinur. J. ACM 2007]

Powering step

Say 3-Coloring $\Sigma = \{R, G, B\}$

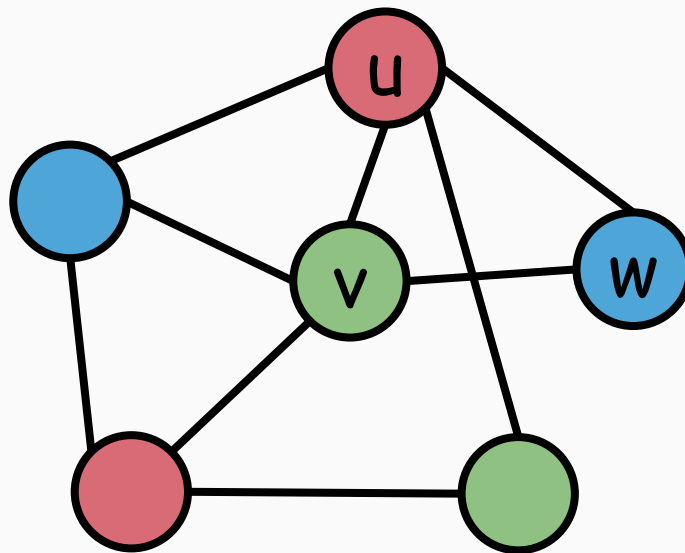
Original $G = (V, E, \Sigma, \Pi = (\pi_e)_{e \in E}) \rightarrow$ New $G' = (V, E', \Sigma', \Pi')$

Asgmt. $\psi: V \rightarrow \Sigma$

\rightarrow Asgmt. $\psi': V \rightarrow \Sigma^V$

for simplicity

⚠ G must be EXPANDER



• $\psi'(x)[v] \stackrel{\text{def}}{=} \text{"opinion" of } \psi'(x) \text{ about the value of } v$

• edge of G' = a length- T random walk over G

const. parameter

Recap: Dinur's gap amplification [Dinur. J. ACM 2007]

Powering step

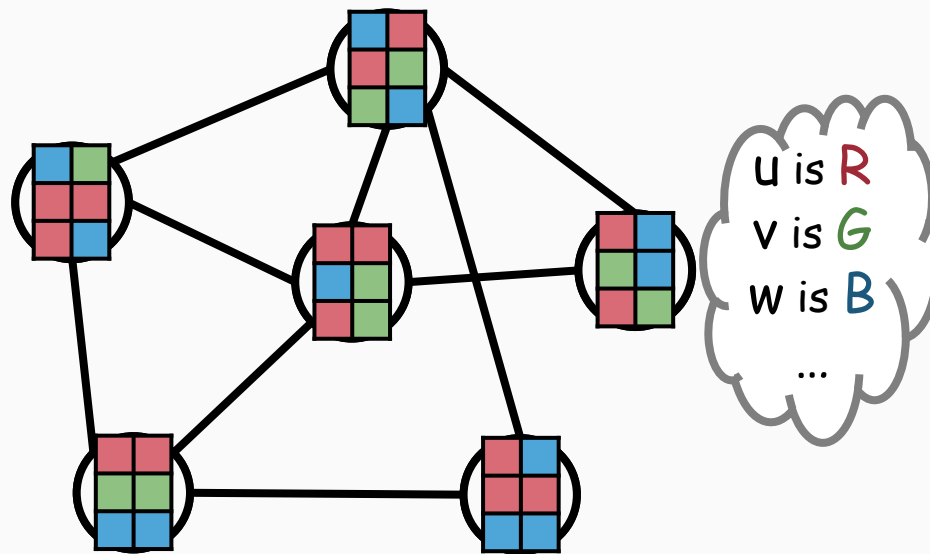
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Powering step

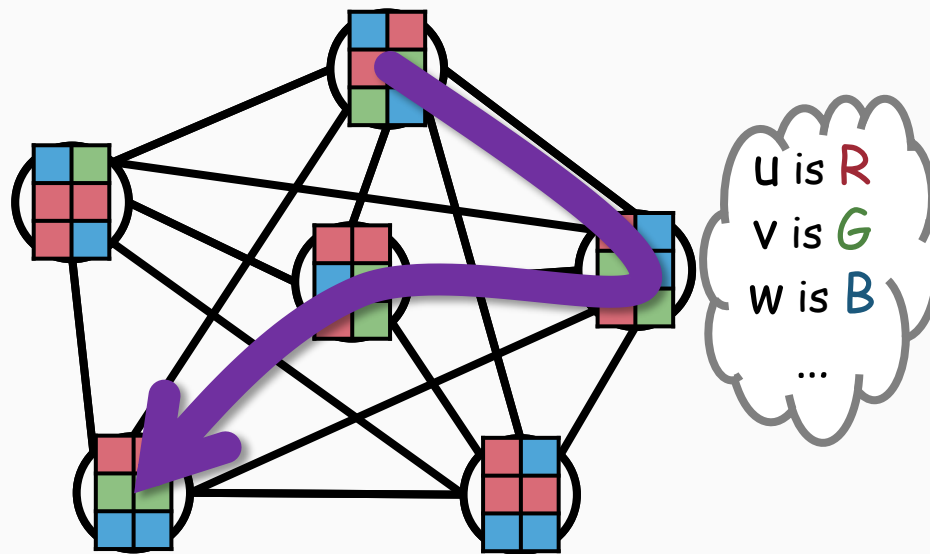
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Recap: Dinur's gap amplification [Dinur. J. ACM 2007]

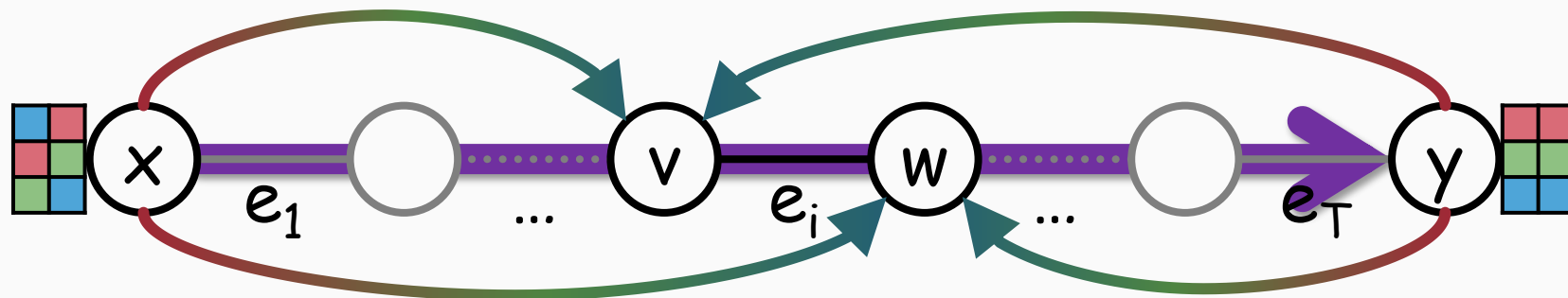
Verifier's test on G' [Radhakrishnan.ICALP 2006]

Pick a random walk $W = \langle e_1, \dots, e_T \rangle$ from x to y

$\psi'(x)$ & $\psi'(y)$ pass the test at $e_i = (v,w)$ if

x & y agree on color of (v,w)
opinions about (v,w) satisfy $\pi_{(v,w)}$

ψ' satisfies π'_W $\stackrel{\text{def}}{\iff}$ $\psi'(x)$ & $\psi'(y)$ pass test at every edge in W



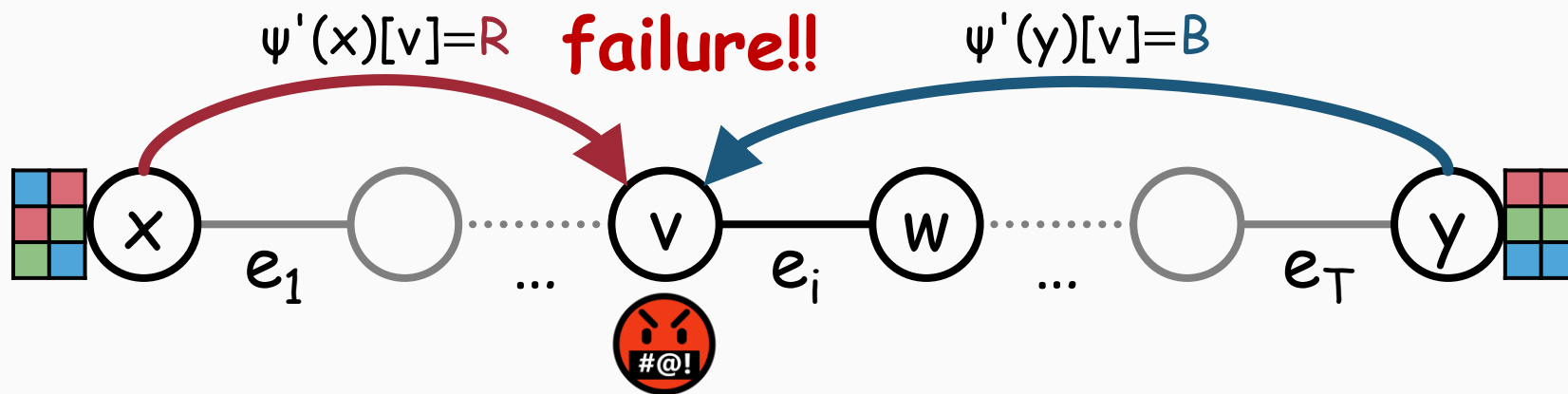
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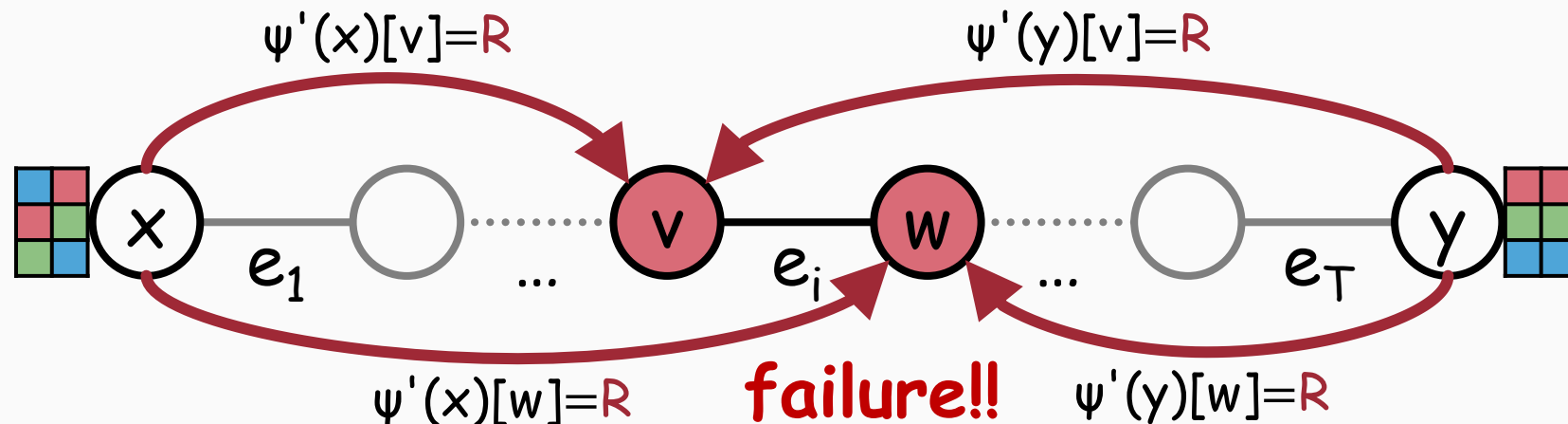
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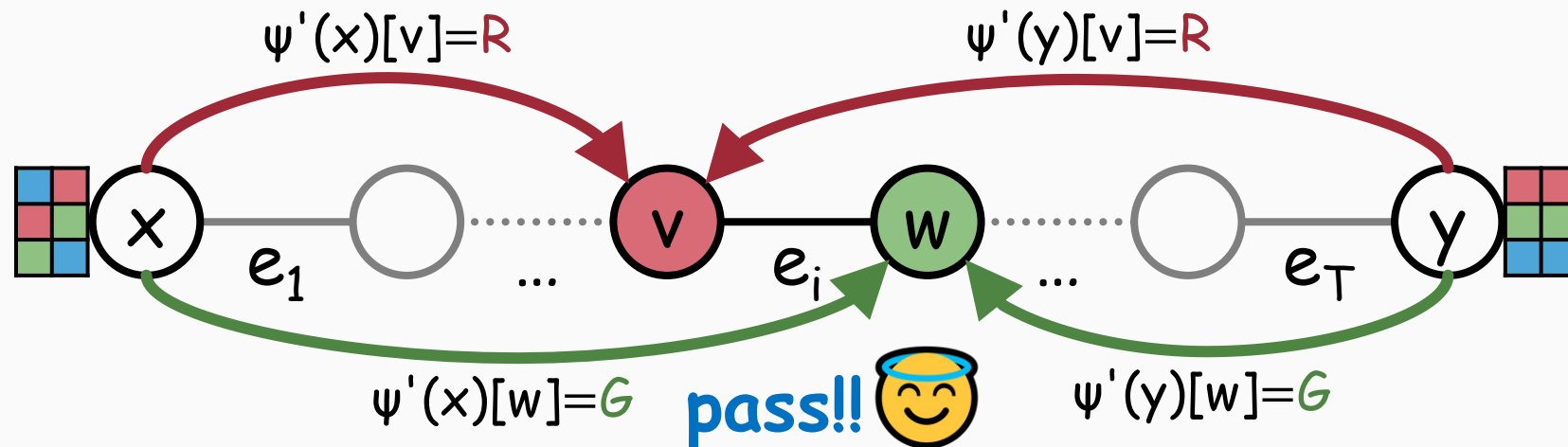
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- $(\psi'(x)[v], \psi'(x)[w])$ satisfies e_i



Recap: **Maxmin** Binary CSP Reconfiguration

[Ito et al. Theor. Comput. Sci. 2011] [O. STACS 2023]

- **Input:** Binary CSP $G = (V, E, \Sigma, \Pi = (\pi_e)_{e \in E})$ & satisfying $\psi_s, \psi_t: V \rightarrow \Sigma$
- **Output:** $\psi = \langle \psi^{(0)} = \psi_s, \dots, \psi^{(\ell)} = \psi_t \rangle$ (reconf. sequence) s.t.
 - ~~ψ satisfies all edges of G~~ (feasibility)
 - $\text{Ham}(\psi^{(i-1)}, \psi^{(i)}) = 1$ (adjacency on hypercube)
- **Goal:** $\max_{\psi} \text{val}_G(\psi) \stackrel{\text{def}}{=} \min_i (\text{frac. of edges satisfied by } \psi^{(i)})$
 $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) \stackrel{\text{def}}{=} \max. \text{ value of } \rightarrow$

Under **RIH**, $\exists \varepsilon$ Gap[1 vs. 1- ε] Binary CSP Reconf. is **PSPACE-hard**:

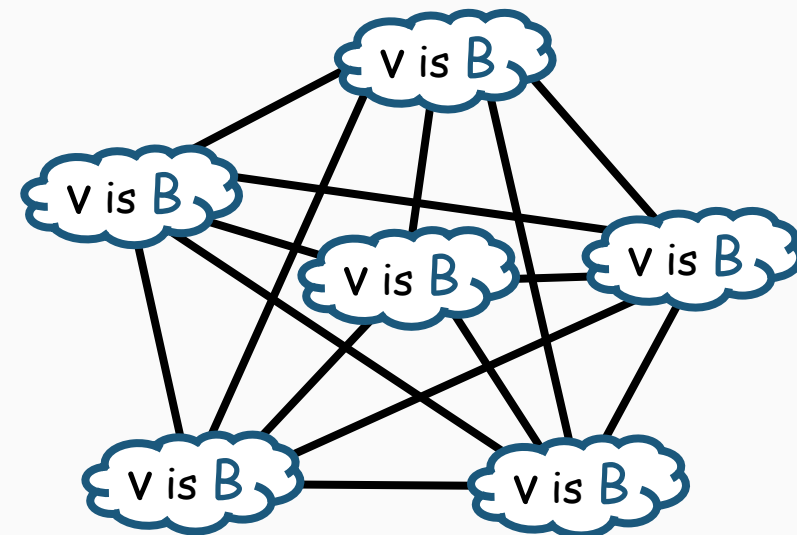
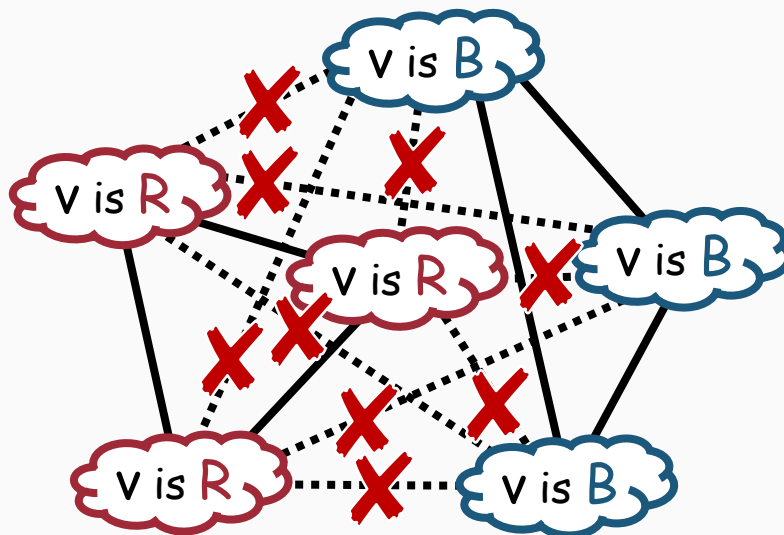
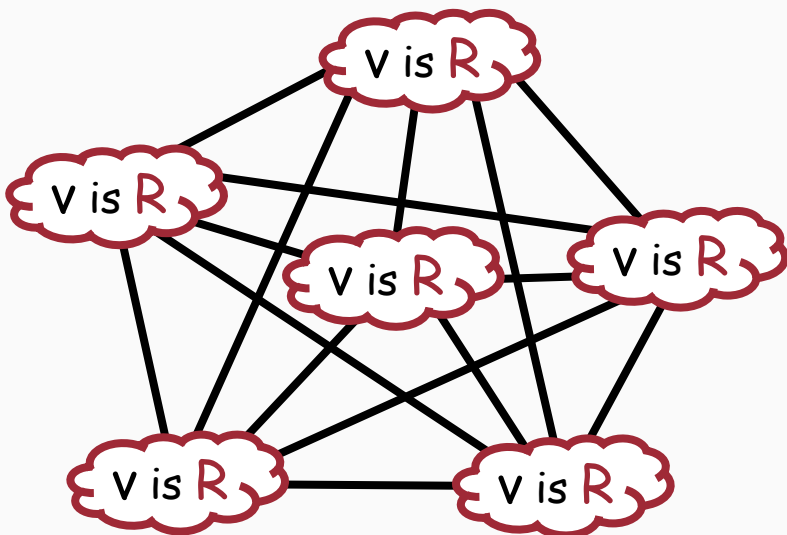
- $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) = 1$ ($\exists \psi$ every $\psi^{(i)}$ satisfies all edges), or
- $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) < 1 - \varepsilon$ ($\forall \psi$ some $\psi^{(i)}$ violates ε -frac. of edges)

Barrier of gap amplification for Binary CSP Reconf.

🥲 Loosing perfect completeness

🎯 Goal: $OPT_G(\psi_s \rightsquigarrow \psi_t) = 1 \not\Rightarrow OPT_{G'}(\psi'_s \rightsquigarrow \psi'_t) = 1$

All vertices should have the SAME opinion about v 's value



$$\forall x \ \psi'_s(x)[v] \stackrel{\text{def}}{=} R$$

$$\exists x, y \ \psi'^{(i)}(x)[v] \neq \psi'^{(i)}(y)[v]$$

$$\forall x \ \psi'_t(x)[v] \stackrel{\text{def}}{=} B$$

🥲 Verifier rejects \rightarrow

Our solution

Alphabet squaring trick [O. STACS 2023]

🎯 Think as if opinion could take a pair of values!

- Original $\Sigma = \{R, G, B\}$
- New $\Sigma_{sq} = \{R, G, B, RG, GB, BR\}$
- a & β are **consistent** $\Leftrightarrow a \subseteq \beta$ or $a \supseteq \beta$

	R	RG	G	GB	B	BR
R	●	●				●
RG	●	●	●			
G		●	●	●		
GB			●	●	●	
B				●	●	●
BR	●				●	●

Our solution

Modifying verifier's test

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R	●	●				●
RG	●	●	●			
G		●	●	●		
GB			●	●	●	
B				●	●	●
BR	●				●	●

Pick RW $W = \langle e_1, \dots, e_T \rangle$ from x to y as before

$\psi'(x)$ & $\psi'(y)$ pass modified test at $e_i = (v,w)$ if

opinions of x & y are **consistent** at (v,w)
opinions about (v,w) satisfy $\pi_{(v,w)}$

Our solution

Modifying verifier's test

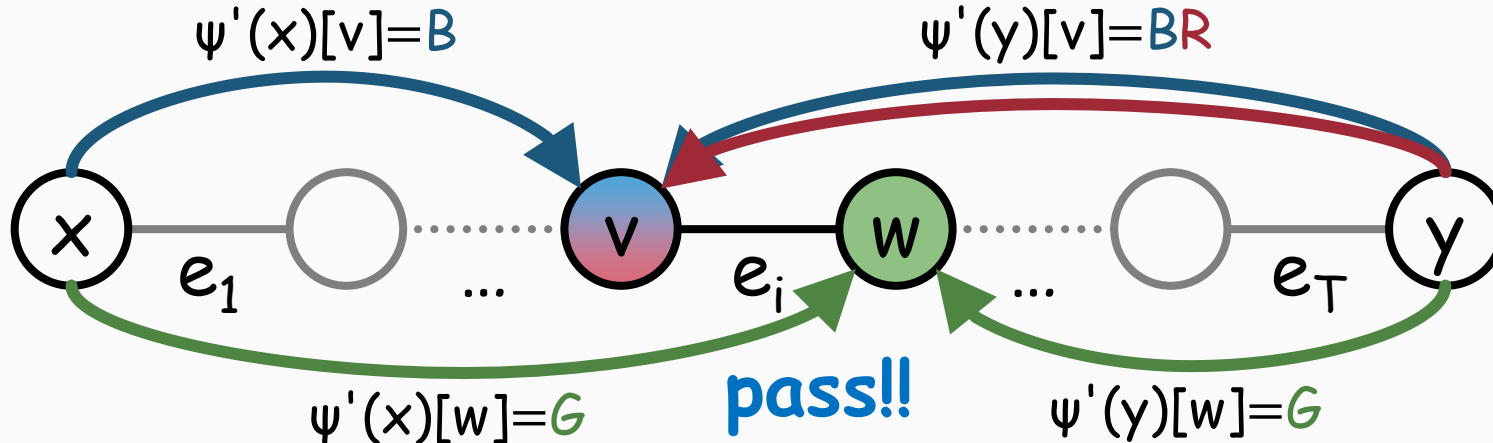
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Our solution

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Pick RW $W = \langle e_1, \dots, e_T \rangle$ from x to y as before

$\psi'(x)$ & $\psi'(y)$ pass modified test at $e_i = (v, w)$ if

(C1) $\psi'(x)[v]$ & $\psi'(y)[v]$ are **consistent**

(C2) $\psi'(x)[w]$ & $\psi'(y)[w]$ are **consistent**

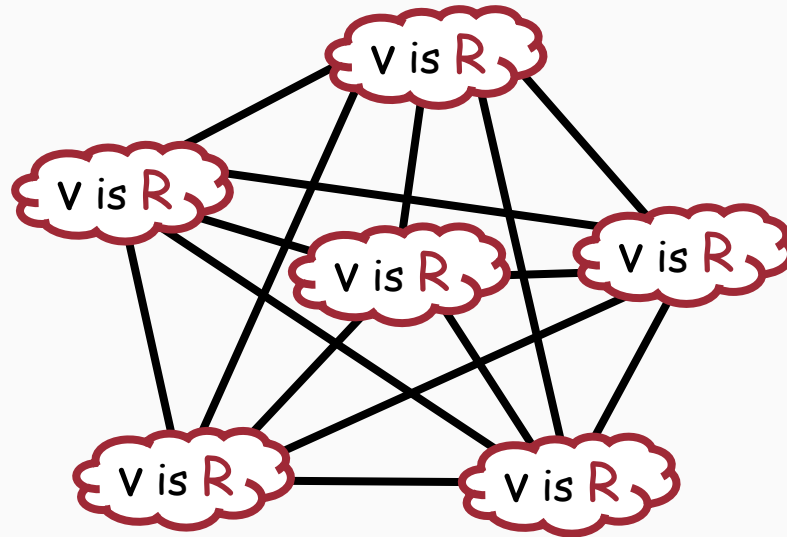
(C3) $(\psi'(x)[v] \cup \psi'(y)[v]) \times (\psi'(x)[w] \cup \psi'(y)[w]) \subseteq \pi_{(v,w)}$

 This verifier is "much weaker" than before

Our solution

😊 Σ_{sq} preserves perfect completeness

🎯 Goal: $OPT_G(\psi_s \leftrightarrow \psi_t) = 1 \implies OPT_{G'}(\psi'_s \leftrightarrow \psi'_t) = 1$

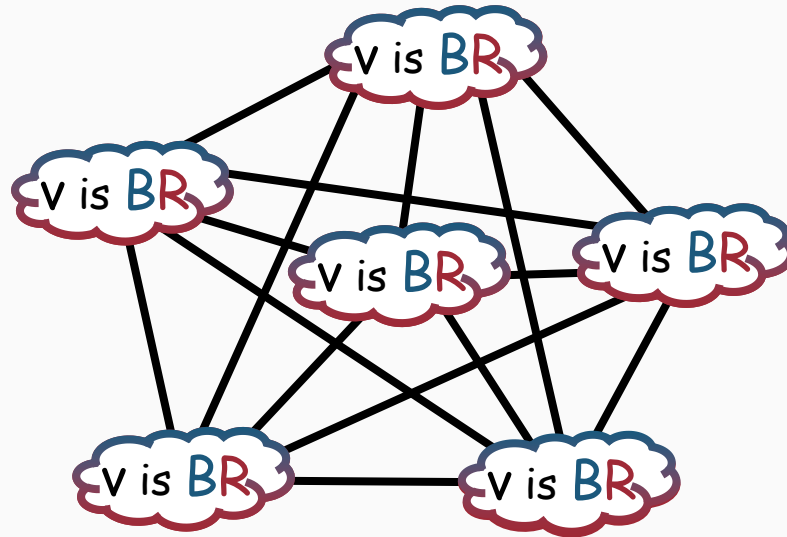


Can transform all **R** opinions into all **B** opinions via **BR**'s

Our solution

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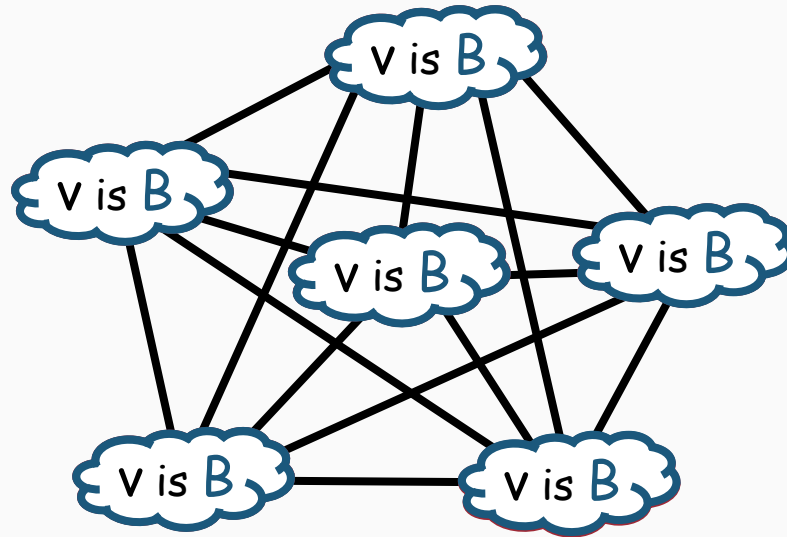


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Our solution

😊 Σ_{sq} preserves perfect completeness


🎯 Goal: $OPT_G(\psi_s \leftrightarrow \psi_t) = 1 \implies OPT_{G'}(\psi'_s \leftrightarrow \psi'_t) = 1$



Can transform all **R** opinions into all **B** opinions via **BR**'s

Our solution

Soundness STILL works

 **Goal:** $OPT_G(\psi_s \rightsquigarrow \psi_t) < 1 - \epsilon \implies OPT_{G'}(\psi'_s \rightsquigarrow \psi'_t) < 1 - \Omega(T \cdot \epsilon)$
 $\psi = \langle \psi^{(0)}, \dots, \psi^{(\ell)} \rangle \longleftarrow \dots$ **plurality vote** $\text{Optimal } \psi' = \langle \psi'^{(0)}, \dots, \psi'^{(\ell)} \rangle$

- We KNOW “ $\exists i \text{ val}_G(\psi^{(i)}) < 1 - \epsilon + o(1)$ ”
- Suppose $\psi^{(i)}$ violates (v, w) of G

$\Pr[\psi^{(i)} \text{ fails modified test at } (v, w) \mid W \text{ touches } (v, w)] = \Omega(1)$

 **DIFFERENT** from
[Radhakrishnan. ICALP 2006]

$\therefore \psi^{(i)}: V \rightarrow (\Sigma_{sq})^V$ but $\psi^{(i)}: V \rightarrow \Sigma$
 $\{R, G, B, RG, GB, BR\}$ $\{R, G, B\}$

Conclusions & open problems

- Gap amplification for Binary CSP Reconf. à la Dinur



- Optimal inapproximability? Maybe $\frac{1}{2}$
- Other gap amplification techniques?
- Alphabet reduction?

Thank you!

