

Gap Preserving Reductions Between Reconfiguration Problems

Naoto Ohsaka



What are reconfiguration problems?

Transform initial state into target state by repeating small changes

12	1	2	15
11	6	5	8
7	10	9	4
	13	14	3



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

- Classical puzzles: 15-puzzles, Rubik's cube, sliding block puzzles
- Understand the structure of solution space applications in dynamic environments
- Unified framework: defined w.r.t. feasibility & adjacency
[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. *Theor. Comput. Sci.*'11]
- Excellent surveys [Nishimura. *Algorithms*'18] [van den Heuvel. '13]

Example

3-SAT Reconfiguration

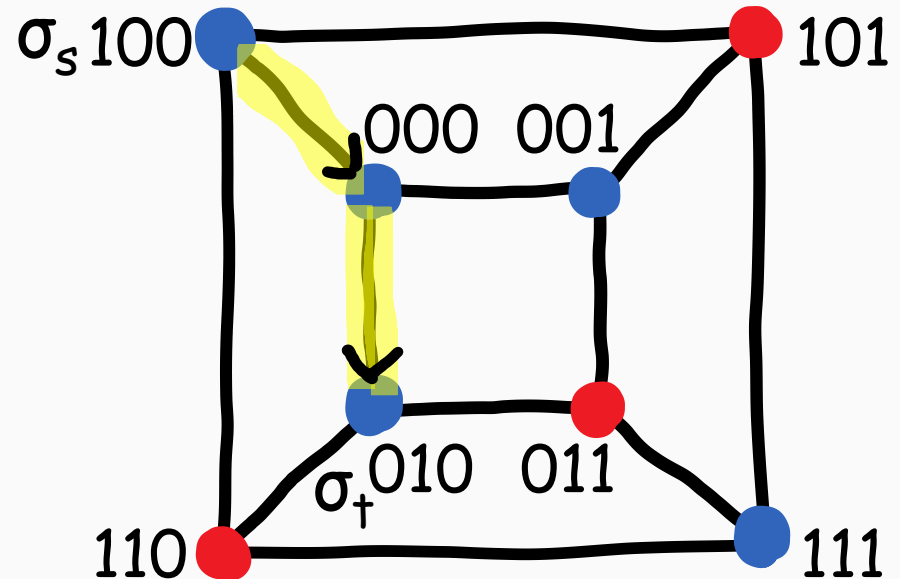
- **Input:** 3-CNF formula φ & satisfying σ_s, σ_t
- **Output:** $\sigma = \langle \sigma^{(0)} = \sigma_s, \dots, \sigma^{(\ell)} = \sigma_t \rangle$ s.t.
 - $\sigma^{(i)}$ satisfies φ (feasibility)
 - $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)

YES case

$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

$$\sigma_s = (1,0,0)$$

$$\sigma_t = (0,1,0)$$



Example

3-SAT Reconfiguration

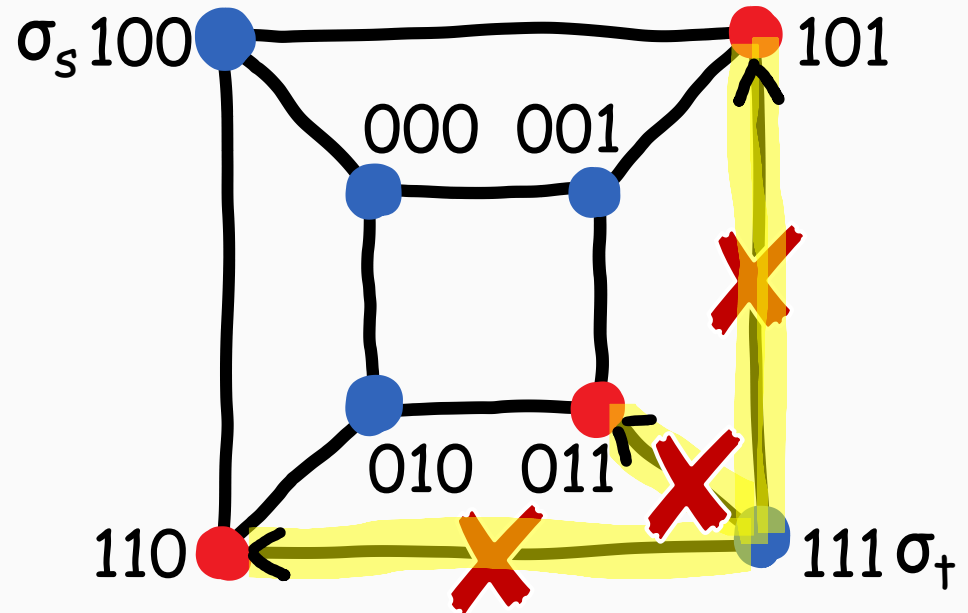
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 - $\sigma^{(i)}$ satisfies φ (feasibility)
 - $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)

NO case

$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

$$\sigma_s = (1,0,0)$$

$$\sigma_t = (1,1,1)$$



Our focus & motivation: Approximate reconfigurability

Even if...

- 😞 **NOT** reconfigurable! and/or
- 😞 many problems are **PSPACE-complete!**

Still want a “reasonable” sequence (quickly)
(e.g.) made up of almost-satisfying assignments



Relax feasibility to obtain *optimization variants*

Example'

Maxmin 3-SAT Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. *Theor. Comput. Sci.*'11]

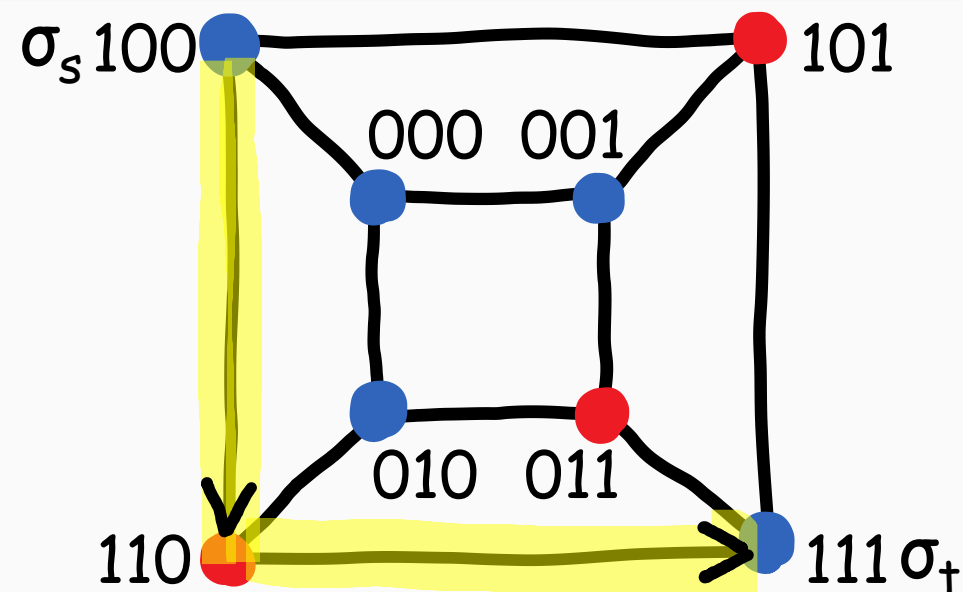
- **Input:** 3-CNF formula φ & satisfying σ_s, σ_t
- **Output:** $\sigma = \langle \sigma^{(0)} = \sigma_s, \dots, \sigma^{(\ell)} = \sigma_t \rangle$ s.t.
 - ~~$\sigma^{(i)}$ satisfies φ~~ (feasibility)
 - $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)
- **Goal:** $\max_{\sigma} \text{val}_{\varphi}(\sigma) \stackrel{\text{def}}{=} \min_i (\text{frac. of satisfied clauses by } \sigma^{(i)})$

$$\varphi = (\bar{x}v\bar{y}vz) \wedge (\bar{x}vyv\bar{z}) \wedge (xv\bar{y}v\bar{z})$$

- $\sigma_s = (1,0,0)$

- $\sigma_t = (1,1,1)$

→ $\text{val}_{\varphi}(\sigma) = 2/3$



Known results on optimization variants

How computationally hard?

Approximability

[Ito+. *Theor. Comput. Sci.* '11] [Ito-Demaine. *J. Comb. Optim.* '14] [O.-Matsuoka. *WSDM* '22]

- Set Cover Reconf., Subset Sum Reconf., Submodular Reconf.

Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

NP-hardness of approximation

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. *Theor. Comput. Sci.* '11]

- SAT Reconfiguration & Clique Reconfiguration

↑ Rely on NP-hardness of combinatorial optimization problems...

Significance of showing PSPACE-hardness...

- no poly-time algorithm ($P \neq PSPACE$)
- no poly-length sequence ($NP \neq PSPACE$)

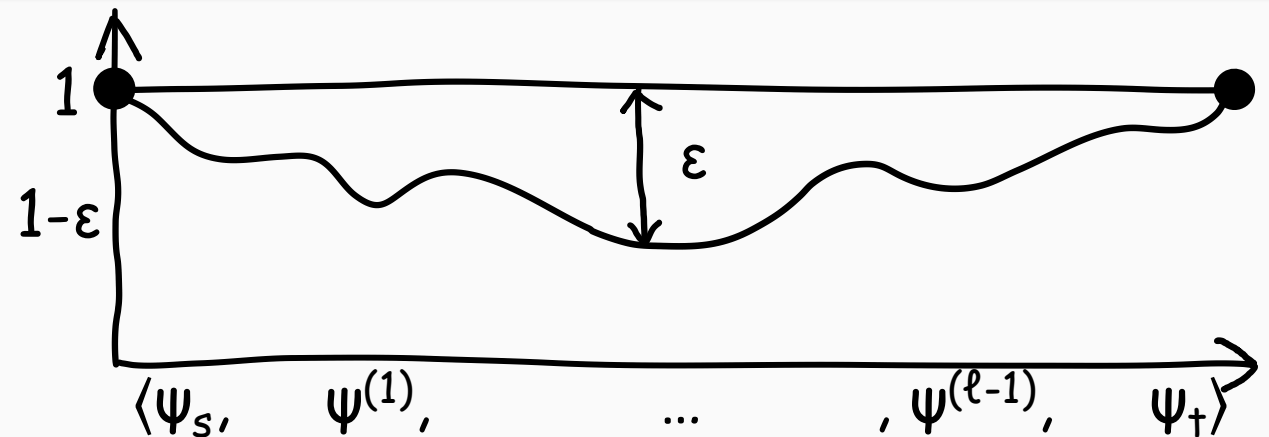
Our working hypothesis & question

Reconfiguration Inapproximability Hypothesis (RIH)

q -ary CSP G & satisfying ψ_s, ψ_t , **PSPACE**-hard to distinguish between

- $\exists \psi \text{ val}_G(\psi) = 1$ (some sequence violates **no** constraint)
- $\forall \psi \text{ val}_G(\psi) < 1 - \varepsilon$ (any sequence violates **> ε -frac.** of constraints)

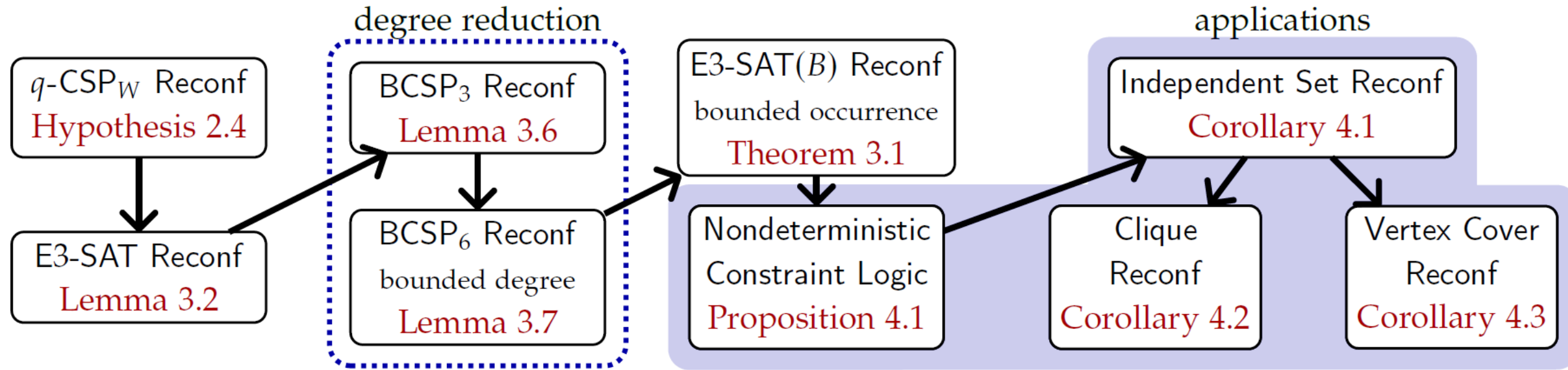
- True if "NP-hard" is used
[Ito+. *Theor. Comput. Sci.* '11]
- Reconfiguration analogue
of the PCP theorem (?)



Q. Which reconfiguration problems are
PSPACE-hard to approximate under (seemingly) plausible **RIH**?

Our results

- Under **RIH**, many problems are **PSPACE**-hard to approximate!!



Sequence of gap-preserving reductions

- 😊 Our reductions preserve **perfect completeness**
- YES instance have a solution for (original) decision version

Main result

Maxmin 3-SAT Reconfiguration of bounded occurrence

- **Input:** 3-CNF formula φ of bounded occurrence & satisfying σ_s, σ_t
- **Output:** $\sigma = \langle \sigma^{(0)} = \sigma_s, \dots, \sigma^{(\ell)} = \sigma_t \rangle$ s.t.
 $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$ (adjacency on hypercube)

Define the value of best sequence:

$$\text{OPT}_{\varphi}(\sigma_s \rightsquigarrow \sigma_t) \stackrel{\text{def}}{=} \max_{\sigma} \min_i (\text{frac. of clauses satisfied by } \sigma^{(i)})$$

Under **RIH**, **PSPACE**-hard to distinguish between

- $\text{OPT}_{\varphi}(\sigma_s \rightsquigarrow \sigma_t) = 1$ ($\exists \sigma$ every $\sigma^{(i)}$ satisfies φ), or
- $\text{OPT}_{\varphi}(\sigma_s \rightsquigarrow \sigma_t) < 1 - \varepsilon$ ($\forall \sigma$ some $\sigma^{(i)}$ violates ε -frac. of clauses)

Most technical step in this paper

Degree reduction of reconfiguration problems

- $\text{Gap}_{1,1-\varepsilon}$ Binary CSP Reconfiguration

Promise: $\text{OPT}_G(\psi_s \leftrightarrow \psi_t) = 1$ or $\text{OPT}_G(\psi_s \leftrightarrow \psi_t) < 1-\varepsilon$

↓ ⚠ Reduction preserving gap & reconfigurability

- $\text{Gap}_{1,1-\varepsilon'}$ Binary CSP Reconfiguration of **max. degree Δ**

Promise: $\text{OPT}_{G'}(\psi'_s \leftrightarrow \psi'_t) = 1$ or $\text{OPT}_{G'}(\psi'_s \leftrightarrow \psi'_t) < 1-\varepsilon'$

😊 Δ & ε' depend only on ε

Why important?

Can reduce to Maxmin 3-SAT Reconfiguration of **bounded occurrence**

In the remainder of this talk...

Proof sketch of degree reduction

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Proof sketch of degree reduction

- Recap of degree reduction of Binary CSP

[Papadimitriou-Yannakakis. *J. Comput. Syst. Sci.* '91] also used by [Dinur. *J. ACM* '07]

- Simple application of \rightarrow to Binary CSP Reconfiguration loses perfect completeness

- **TRICK: Alphabet squaring**

- 😊 Preserves perfect completeness
- 😞 But, NOT a Karp reduction

- Sketching soundness proof

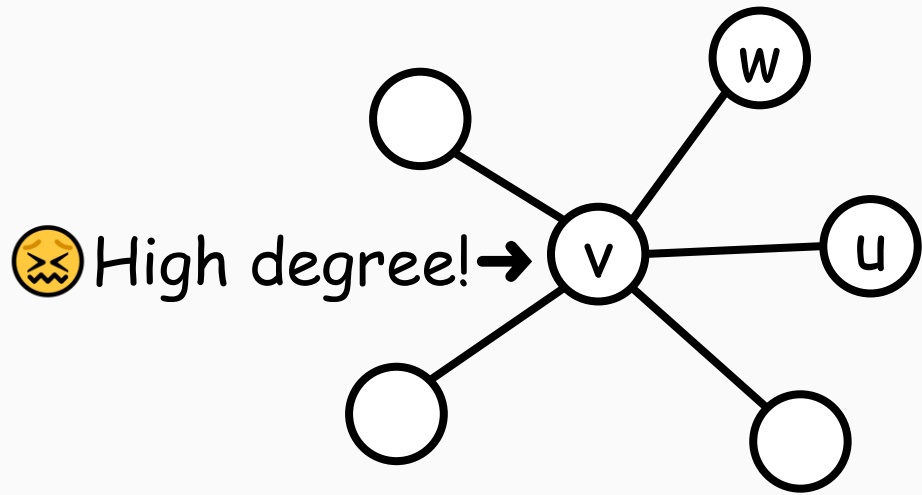
- Why we use expander mixing lemma & near-Ramanujan graphs

Recap of degree reduction of Binary CSP

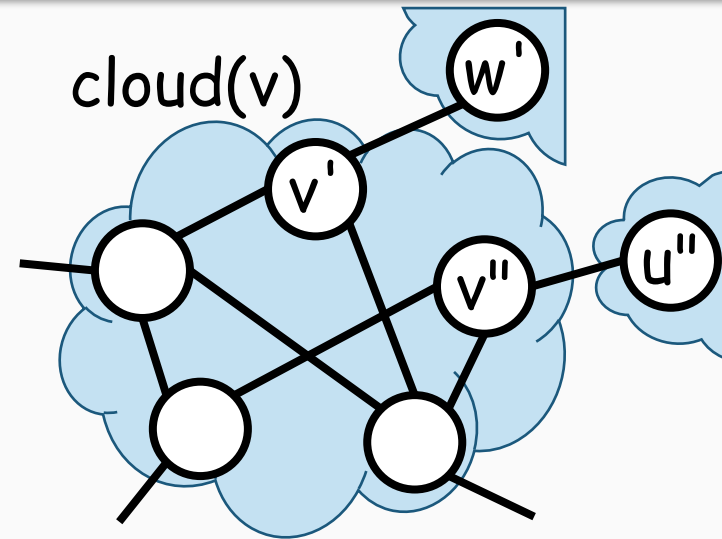
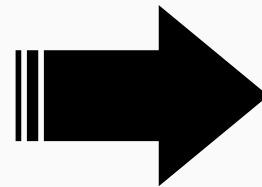
[Papadimitriou-Yannakakis. *J. Comput. Syst. Sci.* '91]

- **Input:** Binary CSP $G=(V,E,\Sigma,\Pi=(\pi_e)_{e\in E})$
- **Output:** $\psi: V\rightarrow\Sigma$
- **Goal:** \max_{ψ} (frac. of edges satisfied by ψ)
 $OPT(G) \stackrel{\text{def}}{=} \text{value of } \nearrow$

ψ satisfies (v,w) if $(\psi(v), \psi(w)) \in \pi_{(v,w)}$



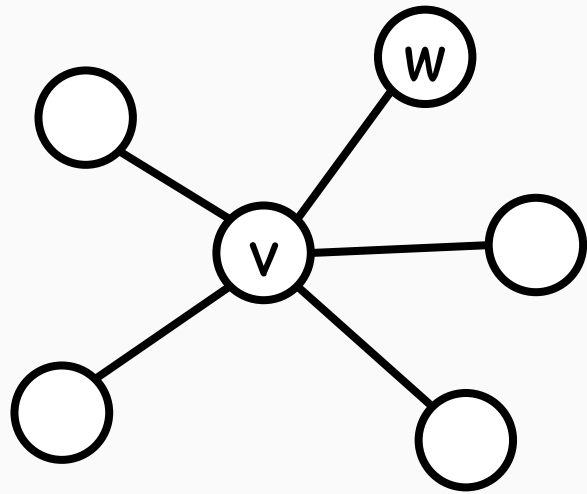
Original $G=(V,E,\Sigma,\Pi)$



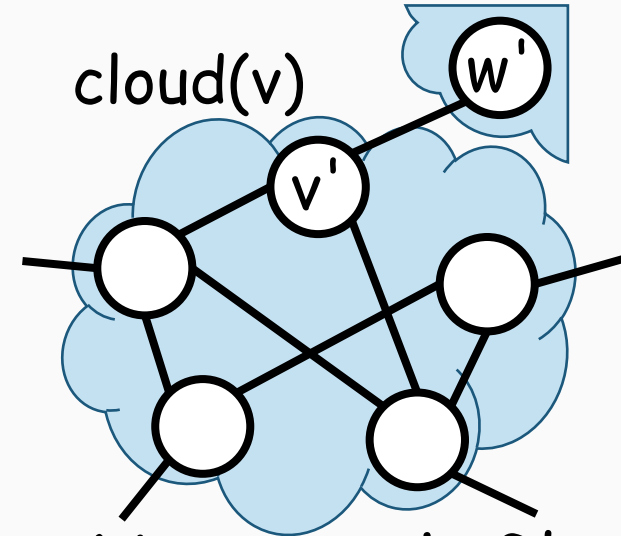
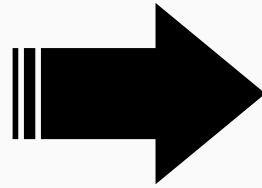
New $G'=(V',E',\Sigma,\Pi')$

Recap of degree reduction of Binary CSP

[Papadimitriou-Yannakakis. *J. Comput. Syst. Sci.* '91]



Original graph G



New graph G'

• Completeness: $OPT(G) = 1 \implies$

• Soundness: $OPT(G) < 1 - \epsilon \implies$

$OPT(G') = 1$

$OPT(G') < 1 - \epsilon'$

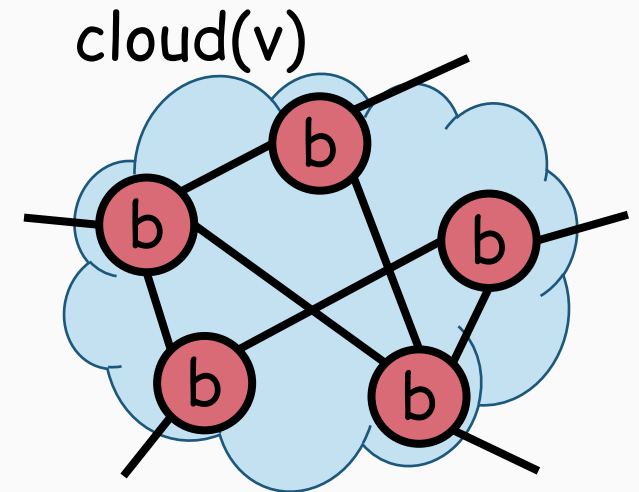
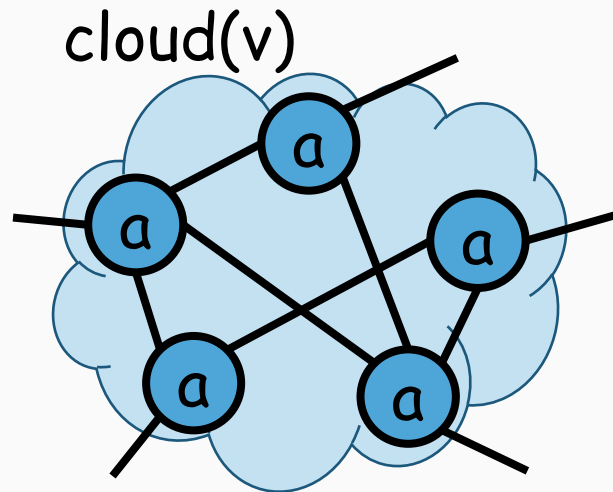
cloud(v) SHOULD behave like a single assignment

- Equality constraints on intra-cloud edges
- cloud(v) should be sparse yet well-connected → 😊 expander graphs

Loosing perfect completeness on Maxmin Binary CSP Reconfiguration

- **Input:** Binary CSP $G=(V,E,\Sigma,\Pi=(\pi_e)_{e\in E})$, satisfying $\psi_s, \psi_t: V\rightarrow\Sigma$
- **Output:** $\psi = \langle \psi^{(0)}=\psi_s, \dots, \psi^{(\ell)}=\psi_t \rangle$ s.t. $\text{Ham}(\psi^{(i-1)}, \psi^{(i)}) = 1$
- **Goal:** $\max_{\psi} \min_i (\text{frac. of edges satisfied by } \psi^{(i)})$
 $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) \stackrel{\text{def}}{=} \text{value of } \rightarrow$

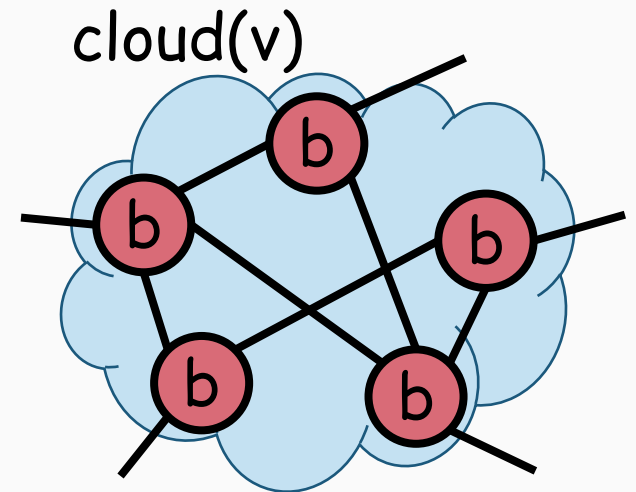
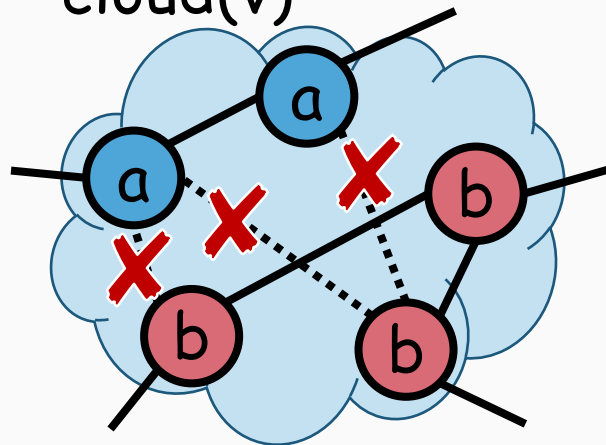
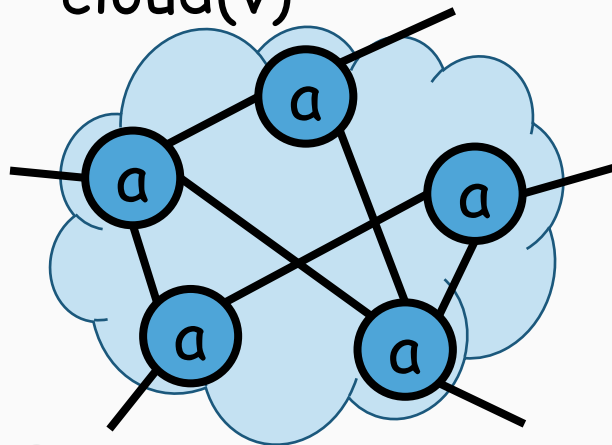
⚠ $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) = 1 \not\Rightarrow \text{OPT}_{G'}(\psi'_s \rightsquigarrow \psi'_t) = 1$



Loosing perfect completeness on Maxmin Binary CSP Reconfiguration

- **Input:** Binary CSP $G=(V,E,\Sigma,\Pi=(\pi_e)_{e\in E})$, satisfying $\psi_s, \psi_t: V\rightarrow\Sigma$
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 $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) \stackrel{\text{def}}{=} \text{value of } \uparrow$

⚠ $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) = 1 \not\Rightarrow \text{OPT}_{G'}(\psi'_s \rightsquigarrow \psi'_t) = 1$



☹ Cannot reconfigure without violating any equality constraints

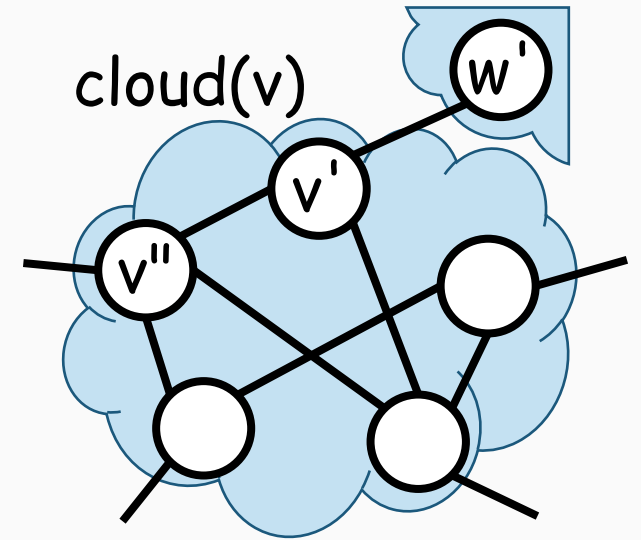
TRICK: Alphabet squaring

🎯 Think as if vertex could take a pair of values!

- Original $\Sigma = \{a, b, c\}$
- New $\Sigma' = \{a, b, c, ab, bc, ca\}$

Constraint for inter-cloud edge $e'=(v',w')$

- Original $\pi_e = \{(a,b), (a,c)\}$
- New $\pi'_{e'} = \{(a,b), (a,c), (a, bc)\}$



Equality-LIKE constraint for intra-cloud edge $e'=(v',v'')$

- $\pi'_{e'} = \{(a, \beta) : a \subseteq \beta \text{ or } \beta \subseteq a\}$
 $= \{(a,a), (b,b), (c,c), (ab,a), (ab,b), (bc,b), (bc,c), (ca,c), (ca,a), (a,ab), (b,ab), (b,bc), (c,bc), (c,ca), (a,ca), (ab,ab), (bc,bc), (ca,ca)\}$

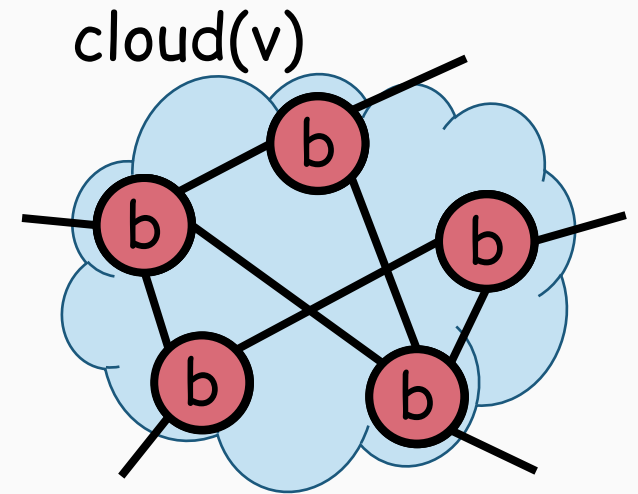
😊 Alphabet squaring preserves perfect completeness

🎯 Think as if vertex could take a pair of values!

- Original $\Sigma = \{a, b, c\}$
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Constraint for inter-cloud edge $e'=(v',w')$

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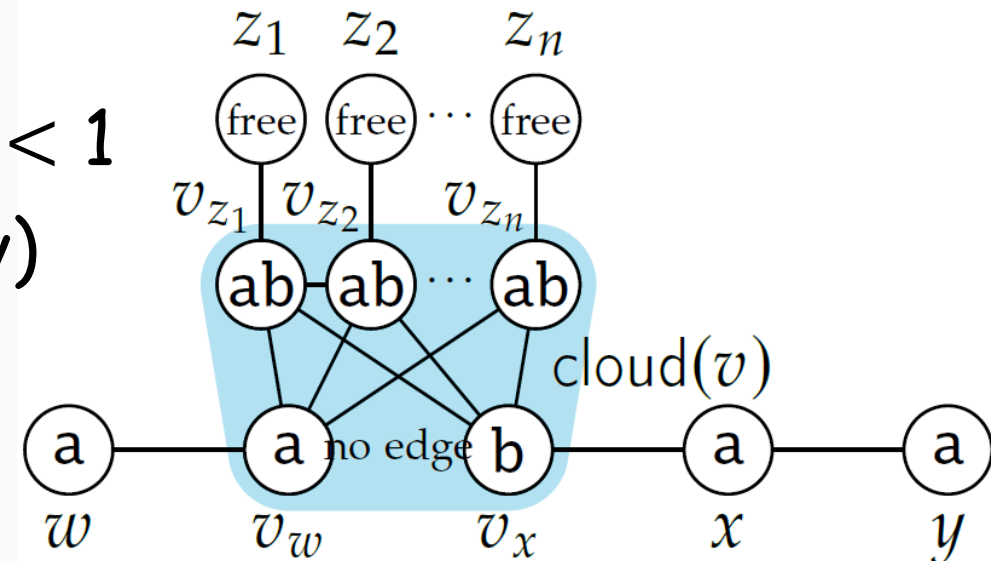
😭 Alphabet squaring is **NOT** a Karp reduction

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- **Output:** $\psi = \langle \psi^{(0)}=\psi_s, \dots, \psi^{(\ell)}=\psi_t \rangle$ s.t. $\text{Ham}(\sigma^{(i-1)}, \sigma^{(i)}) = 1$
- **Goal:** $\max_{\psi} \min_i (\text{frac. of edges satisfied by } \psi^{(i)})$
 $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) \stackrel{\text{def}}{=} \text{value of } \curvearrowright$

Apply degree reduction with AS

⚠️ $\text{OPT}_G(\psi_s \rightsquigarrow \psi_t) < 1 \not\Rightarrow \text{OPT}_{G'}(\psi'_s \rightsquigarrow \psi'_t) < 1$

- \therefore Can assign conflicting values to $\text{cloud}(v)$ without sacrificing any constraint
 (See my paper for concrete example \rightarrow)



Sketch of soundness proof

Observation

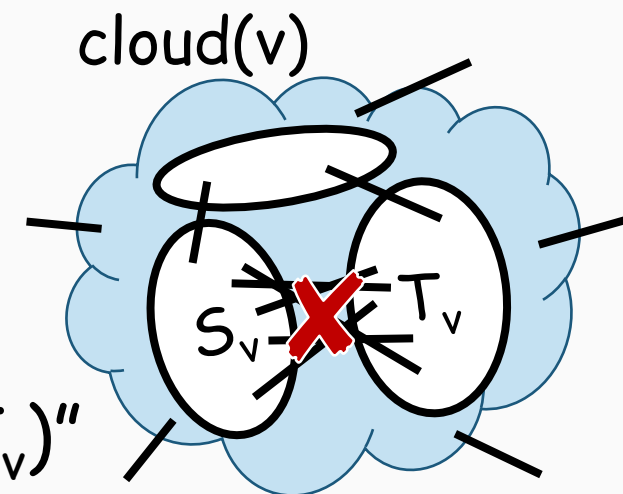
🎯 **Goal:** $OPT_G(\psi_s \leftrightarrow \psi_t) < 1-\epsilon \implies OPT_{G'}(\psi'_s \leftrightarrow \psi'_t) < 1-\epsilon'$
 $\psi = \langle \psi^{(0)}, \dots, \psi^{(\ell)} \rangle \quad \leftarrow \dots \quad \text{Optimal } \psi' = \langle \psi'^{(0)}, \dots, \psi'^{(\ell)} \rangle$
 plurality vote

- Let $\psi \stackrel{\text{def}}{=} \psi^{(i)}$ & $\psi' \stackrel{\text{def}}{=} \psi'^{(i)}$ s.t. " $\psi^{(i)}$ is the WORST assignment"

$D_v \stackrel{\text{def}}{=} \{\text{vertices of cloud}(v) \text{ disagreeing } \psi(v)\}$

- Can assume $\sum_v |D_v| = O(\epsilon |E|)$

similarly to [Papadimitriou-Yannakakis. *J. Comput. Syst. Sci.*'91]



😊 $\exists S_v$ & T_v of size $O(|D_v|)$ s.t. " ψ' violates all $E(S_v, T_v)$ "

Sketch of soundness proof

Bounding # of (violated) edges

- How large is $E(S_v, T_v)$? → **Expander mixing lemma!**
[Alon-Chung. *Discret. Math.* '88]

$$|E(S_v, T_v)| \geq \frac{\text{degree of cloud}(v) \cdot |S_v| \cdot |T_v|}{|\text{cloud}(v)|} - \lambda \sqrt{|S_v| \cdot |T_v|}$$

2nd eigval of cloud(v)
 $\lambda \sqrt{|S_v| \cdot |T_v|}$
⚠ Should be small

- Get $\lambda/d \ll \varepsilon$ by setting $\lambda \approx \varepsilon^{-1}$ & $d \approx \varepsilon^{-2}$
 using **near-Ramanujan graphs**
[Alon. *Comb.* '21]
[Mohanty-O'Donnell-Paredes. *SIAM J. Comput.* '21]
- If $|D_v| = O(\varepsilon \cdot \text{cloud}(v))$ (hold for many v)

😊 $|E(S_v, T_v)| = O(|D_v|)$

Taking sum over v (of large D_v) derives
 (total frac. of violated edges in G') $> \varepsilon' > 0$

Conclusion and future work

Combinatorial reconfiguration \times Hardness of approximation

🤪 "Is Reconfiguration Inapproximability Hypothesis true...?"

- Use gap amplification? [Dinur. *J. ACM*'07]
- Reduce from **PSPACE**-hard inapproximable problems?
[Condon-Feigenbaum-Lund-Shor. *Chic. J. Theor. Comput. Sci.*'95]
- Adapt a Karp reduction from TQBF
to Nondeterministic Constraint Logic?
[Hearn-Demaine. *Theor. Comput. Sci.*'05]
- Even if false, NP-hardness of approximation

Thank you!



