2024.10.8 5th Combinatorial Reconfiguration Workshop @ Fukuoka, Japan

On the Complexity of **Approximating** Reconfiguration Problems

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Joint work with **Shuichi Hirahara** (NII)

Prelude

When I started studying reconfiguration …

Many nice papers & surveys are available

Prelude

What I was interested in

On the complexity of reconfiguration problems [ISAAC 2008 & Theor. Comput. Sci. 2011]

Takehiro Ito[a,*], Erik D. Demaine[b], Nicholas J.A. Harvey[c], Christos H. Papadimitriou[d], Martha Sideri[e], Ryuhei Uehara[f], Yushi Uno[g]

5. Open problems

There are many open problems raised by this work, and we mention some of these below:

- Can the MATCHING RECONFIGURATION problem for edge-weighted graphs be solved also in polynomial time? We conjecture that the answer is positive.
- Is the TRAVELING SALESMAN RECONFIGURATION problem (where two tours are adjacent if they differ in two edges) PSPACE-complete?
- Are there better approximation algorithms for the MINMAX POWER SUPPLY RECONFIGURATION problem? Lower bounds?
- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

Theme of this talk

This open problem has been resolved

Outline of this talk

Part I

What is meant by "approximation"

Part II

Complexity of approximating reconf. problems

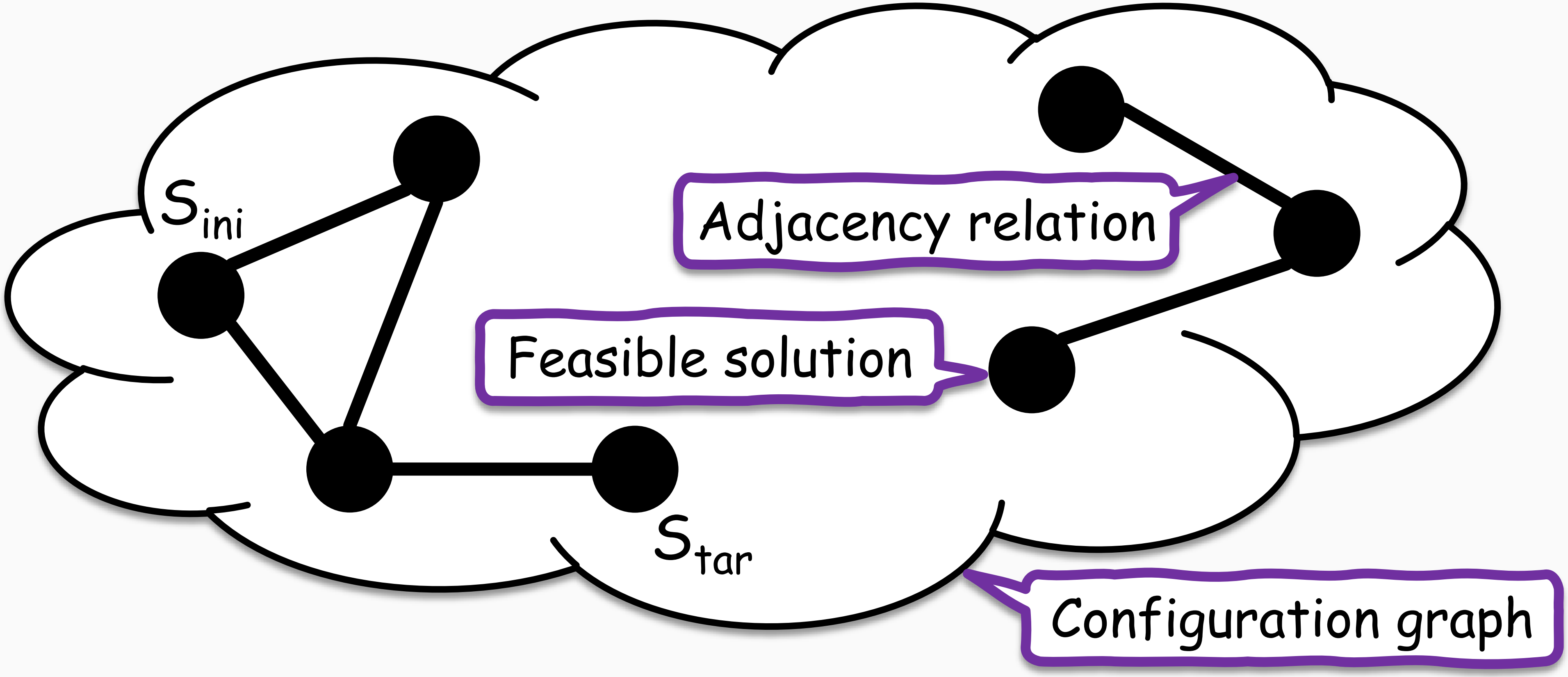
Part III

Recent progress, Struggles & future directions

Recap

Exact reconfiguration — a decision problem

Q. Is a pair of feasible solutions reachable to each other?



Recap

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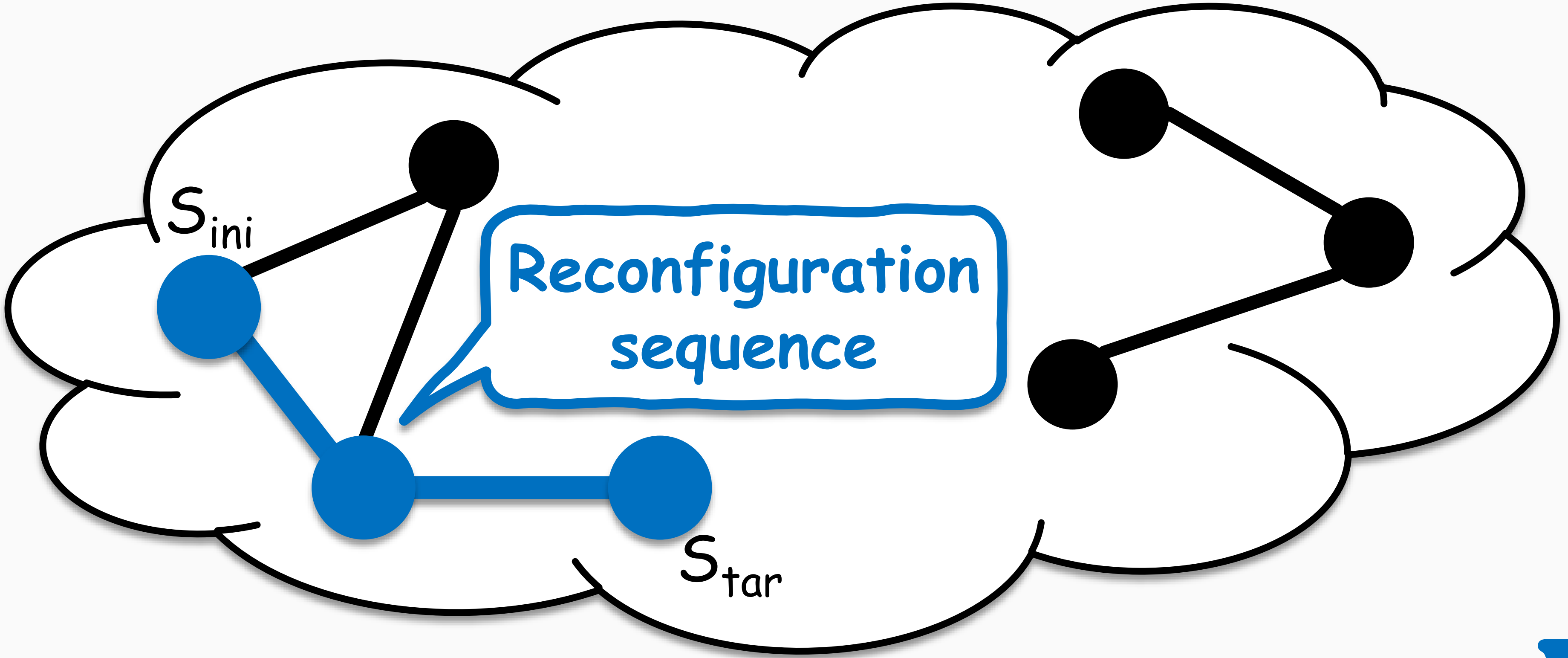


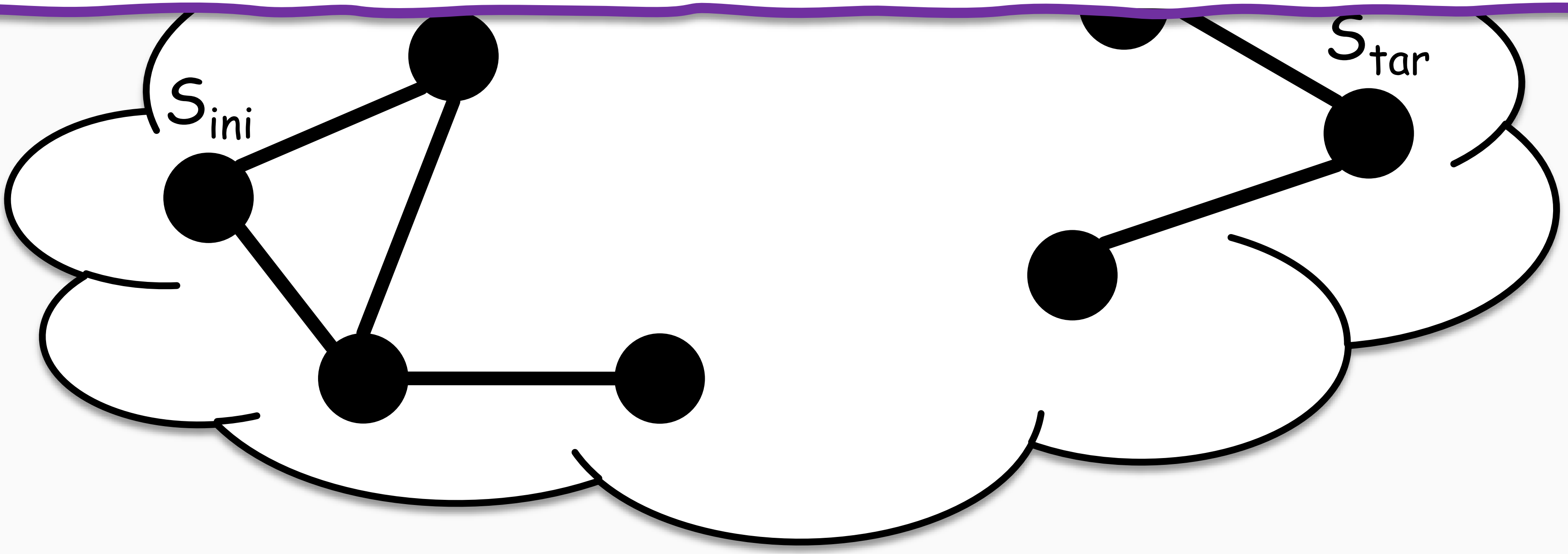
YES

Recap

Exact reconfiguration — a decision problem

🎯Something that "looks" like a reconf. sequence even if

1. we are given a **NO** instance, or
2. we are dealing with intractable reconf. problems

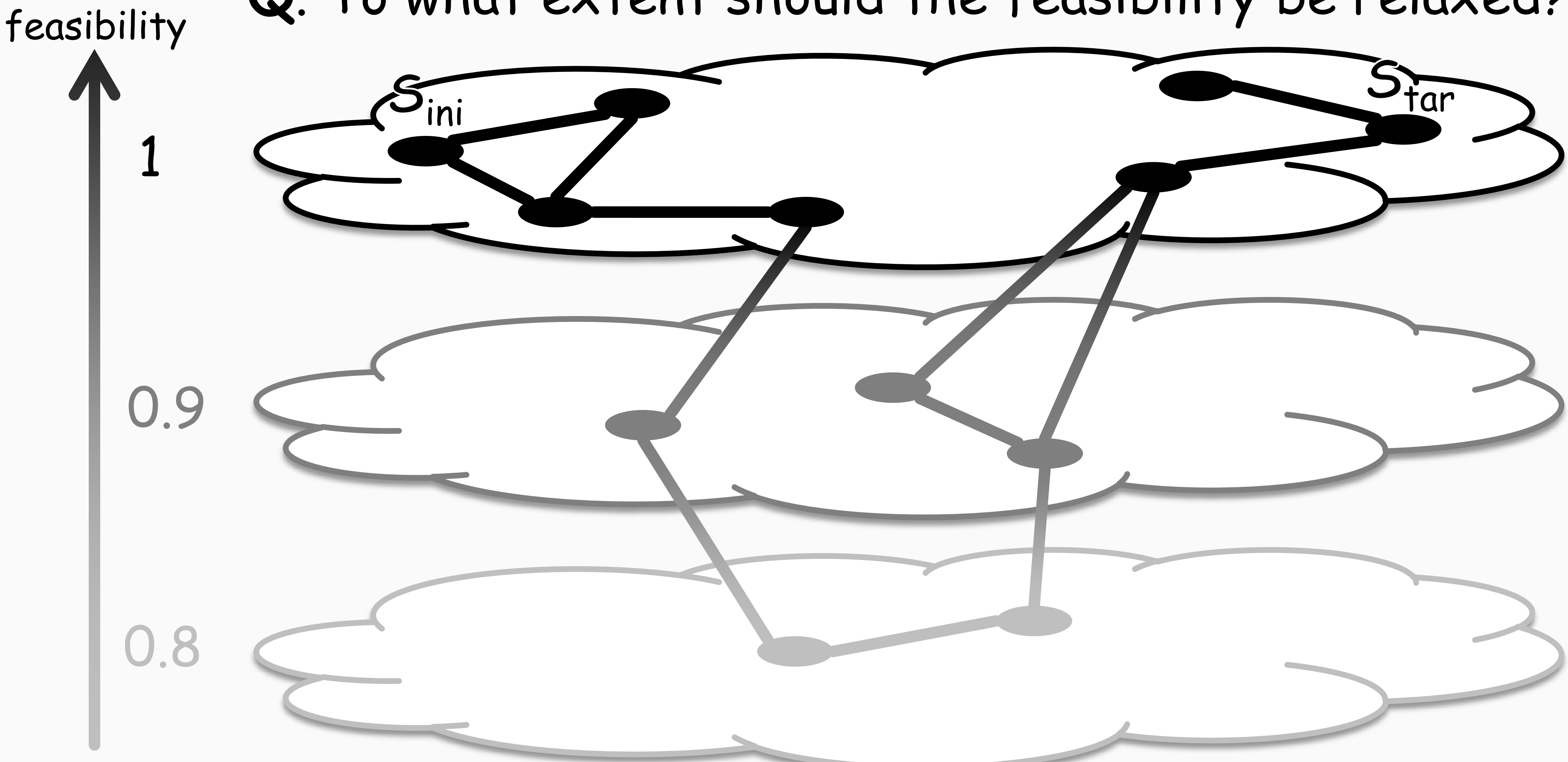


NO

Our focus

Approximate reconf. — an optimization problem

Q. To what extent should the feasibility be relaxed?



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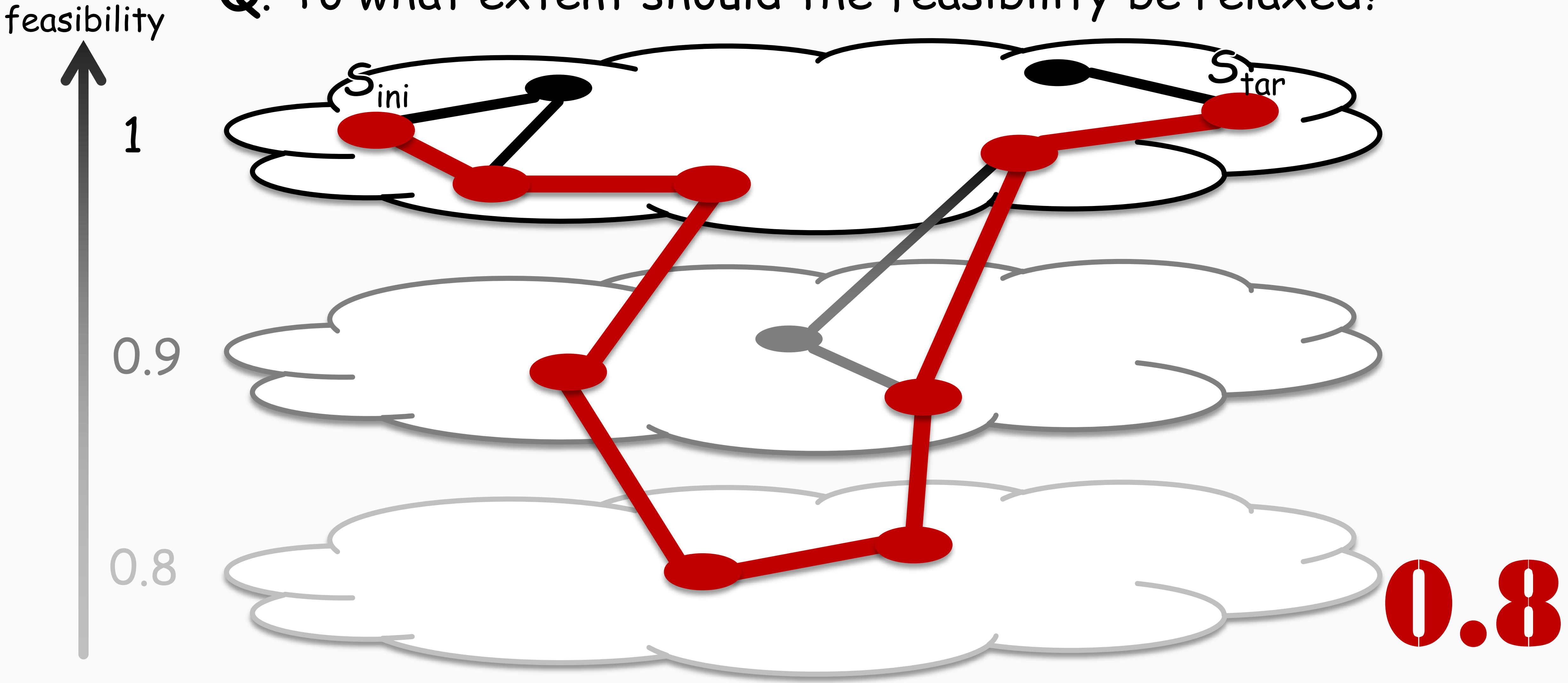

Our focus

Approximate reconf. — an optimization problem

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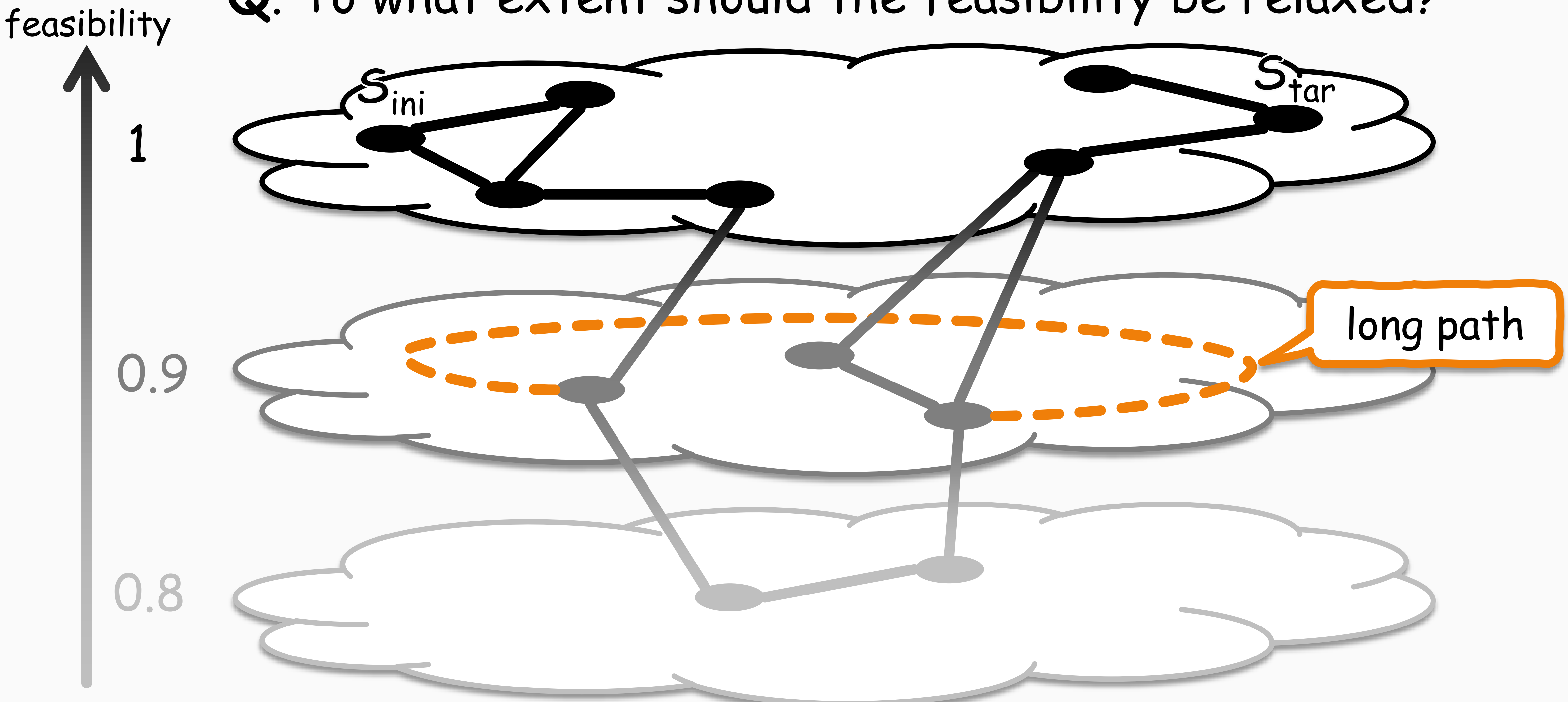

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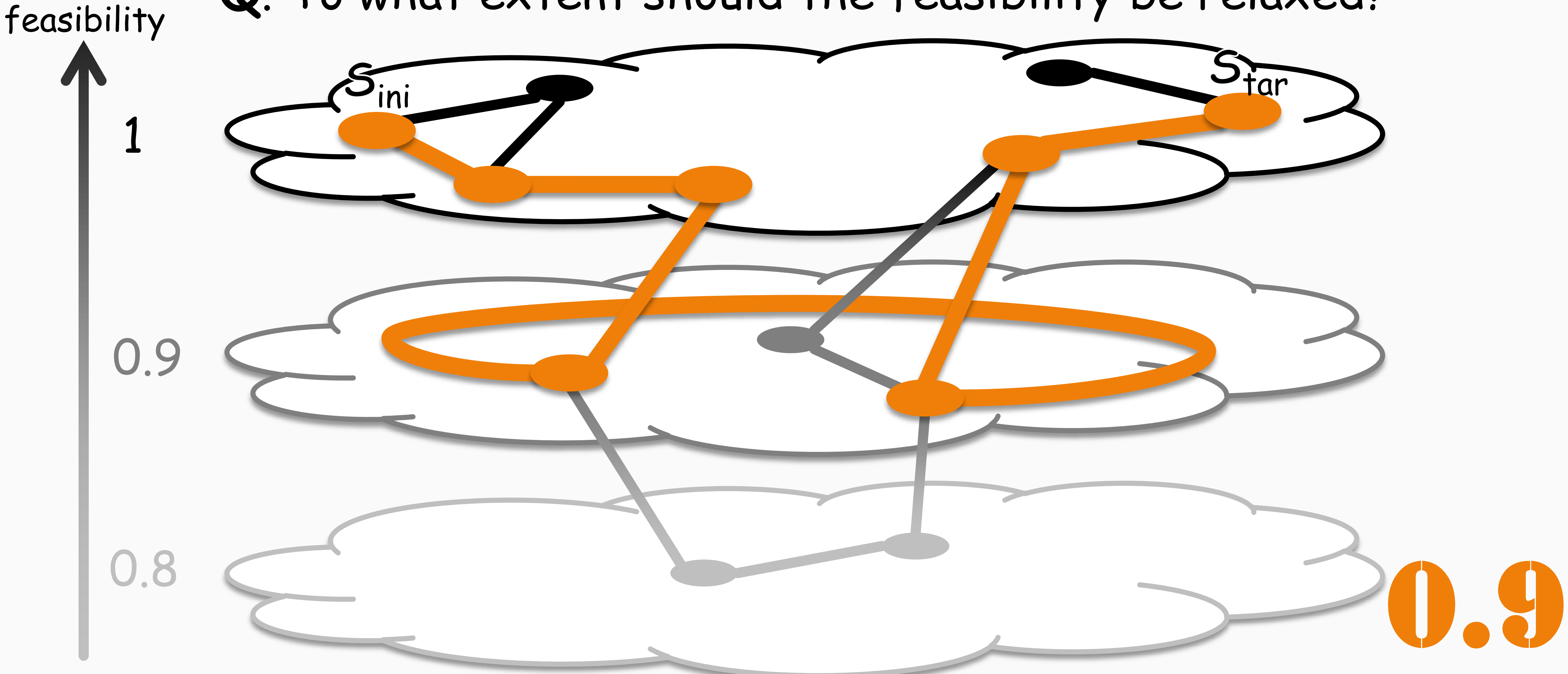


Example 1

3-SAT Reconfiguration

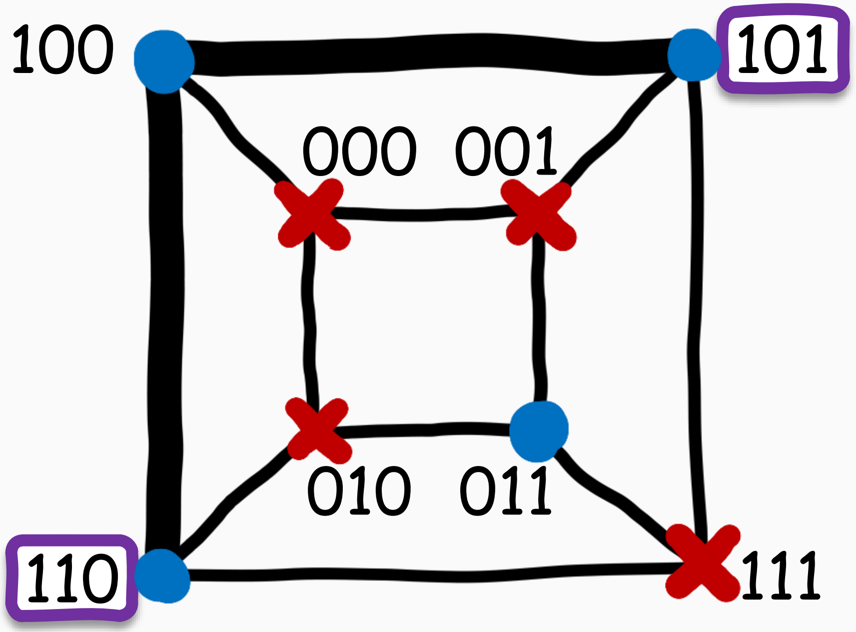
[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
  $\sigma^{(t)}$ satisfies φ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)

$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,1,0)$

$\sigma_{tar}(x,y,z) = (1,0,1)$



YES

Example 2

3-SAT Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
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$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,0,0)$

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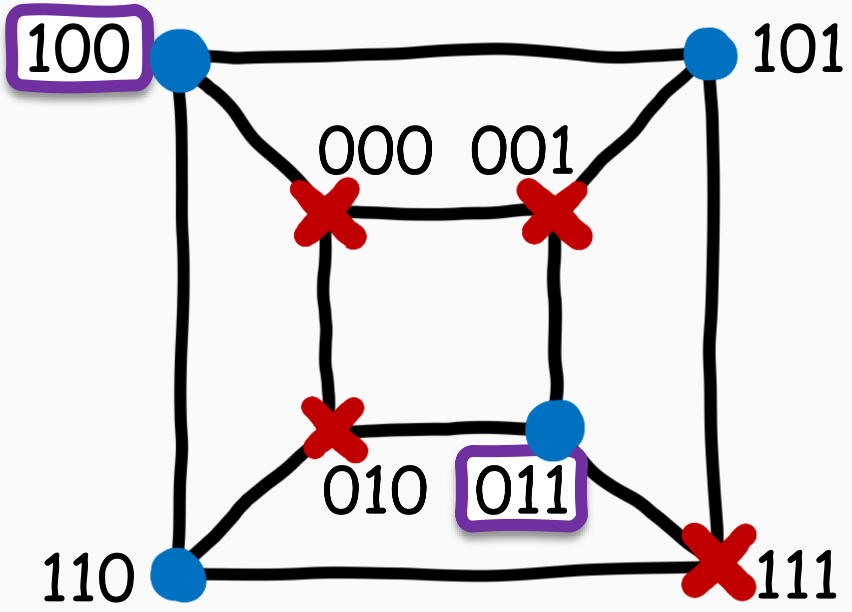


NO

Example 3

Maxmin 3-SAT Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

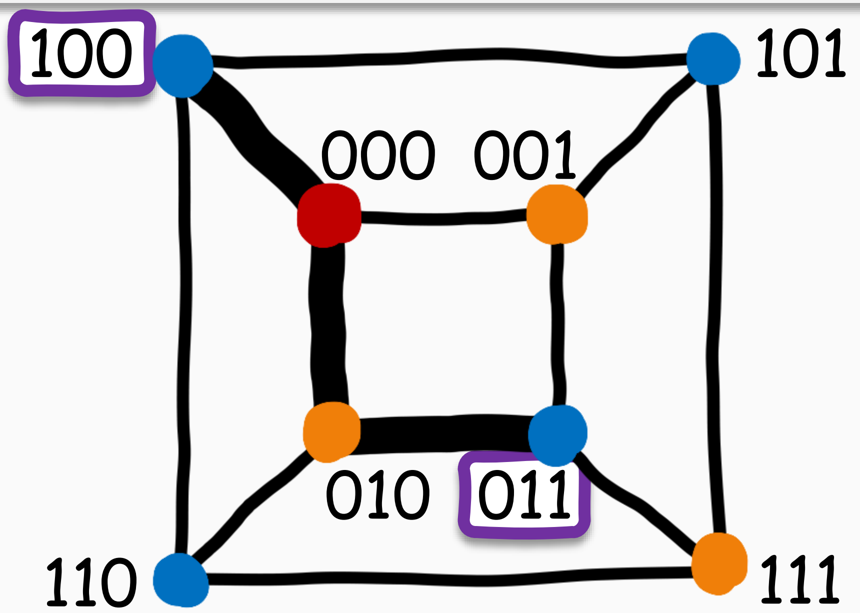
- **Input**: 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output**: $\vec{\sigma} = (\sigma^{(1)}{=}\sigma_{ini}, \ldots, \sigma^{(T)}{=}\sigma_{tar})$ (reconf. sequence) s.t.
  ~~$\sigma^{(t)}$ satisfies φ~~ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)
- **Goal**: maximize $\mathrm{val}_\varphi(\vec{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)

$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,0,0)$

$\sigma_{tar}(x,y,z) = (0,1,1)$

$\mathrm{val}_\varphi(\vec{\sigma}) = \min\{1, \frac{1}{3}, \frac{2}{3}, 1\} = \frac{1}{3}$



0.33

Example 4

Maxmin 3-SAT Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

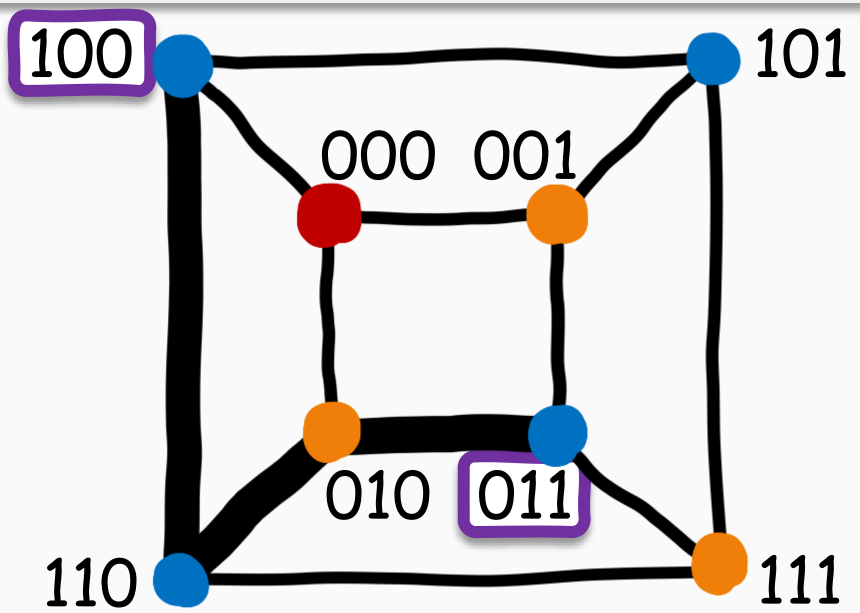
- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
  ~~$\sigma^{(t)}$ satisfies φ~~ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)
- **Goal:** maximize $\mathrm{val}_\varphi(\vec{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)

$\varphi = (x \lor y) \land (x \lor z) \land (\bar{x} \lor \bar{y} \lor \bar{z})$

$\sigma_{ini}(x,y,z) = (1,0,0)$

$\sigma_{tar}(x,y,z) = (0,1,1)$

$\mathrm{val}_\varphi(\vec{\sigma}) = \min\{1, 1, \frac{2}{3}, 1\} = \frac{2}{3}$

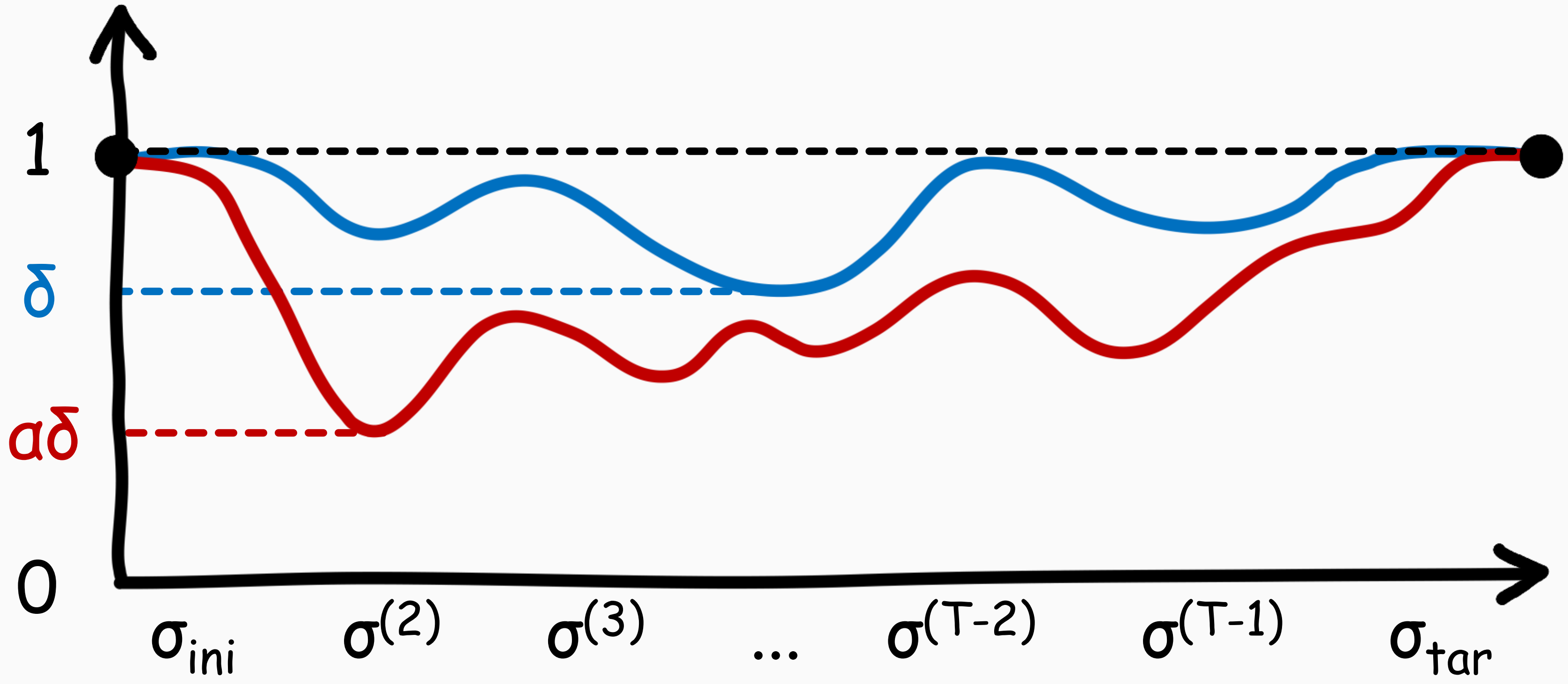


0.66

Defining approximation algorithms

α-approximation algorithm $\mathcal{A}$ for Maxmin 3-SAT Reconf.

- If there is $\vec{\sigma}^*$ s.t. $\mathrm{val}_\varphi(\vec{\sigma}^*) \geq \delta$ ($\forall \sigma^{*(t)}$ satisfies ≥ δ-frac. of clauses)
- then $\mathcal{A}$ finds $\vec{\sigma}$ s.t. $\mathrm{val}_\varphi(\vec{\sigma}) \geq \alpha\delta$ ($\forall \sigma^{(t)}$ satisfies ≥ αδ-frac. of clauses)



Exercise 1

0.5-approx. alg. for Maxmin 3-SAT Reconf.

- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Run:** Sample a random ordering π of vars s.t. $\sigma_{ini}(x) \neq \sigma_{tar}(x)$
  Create a reconf. sequence $\vec{\boldsymbol{\sigma}} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$
  by flipping $\pi(1)$, $\pi(2)$, $\pi(3)$, ...

Observe:

For any clause C satisfied by σ_{ini} & σ_{tar}

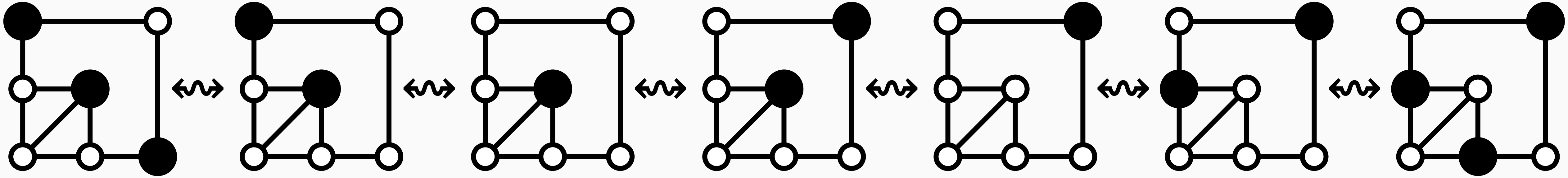
$$\mathbf{Pr}_{\pi}[\text{every } \sigma^{(t)} \text{ satisfies } C] \geq 0.5$$

$$\rightarrow \mathbf{E}_{\pi}[\mathrm{val}_{\varphi}(\vec{\boldsymbol{\sigma}})] \geq 0.5$$

	σ_{ini}	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(4)}$	σ_{tar}
$\pi(3)$	1	1	1	0	0
	1	1	1	1	1
$\pi(1)$	0	1	1	1	1
$\pi(4)$	1	1	1	1	0
	0	0	0	0	0
$\pi(2)$	1	1	0	0	0

Other approximate versions

- **Maxmin Independent Set Reconf.** under token-addition-removal model
  [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
  [de Berg-Jansen-Mukherjee. Discret. Appl. Math. 2018]



- Minmax Vertex Cover Reconf. [Ito-Nooka-Zhou. IEICE Trans. Inf. Syst. 2016]
- Minmax Set Cover Reconf.
  [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
- Subset Sum Reconf. [Ito-Demaine. J. Comb. Optim. 2014]
- Submodular Reconf. [O.-Matsuoka. WSDM 2022]
- Maxmin 2-CSP Reconf. [Karthik C. S.-Manurangsi. 2023] [O. 2023]

Questions of interest

<u>Algorithmic side</u>

- How well can we approximate reconfiguration problems?

<u>Hardness side</u>

- How hard is it to approximate reconfiguration problems?

👆**My interest**

Outline of this talk

Part I

What is meant by "approximation"

Part II

Complexity of approximating reconf. problems

Part III

Recent progress, Struggles & future directions

Exercise 2

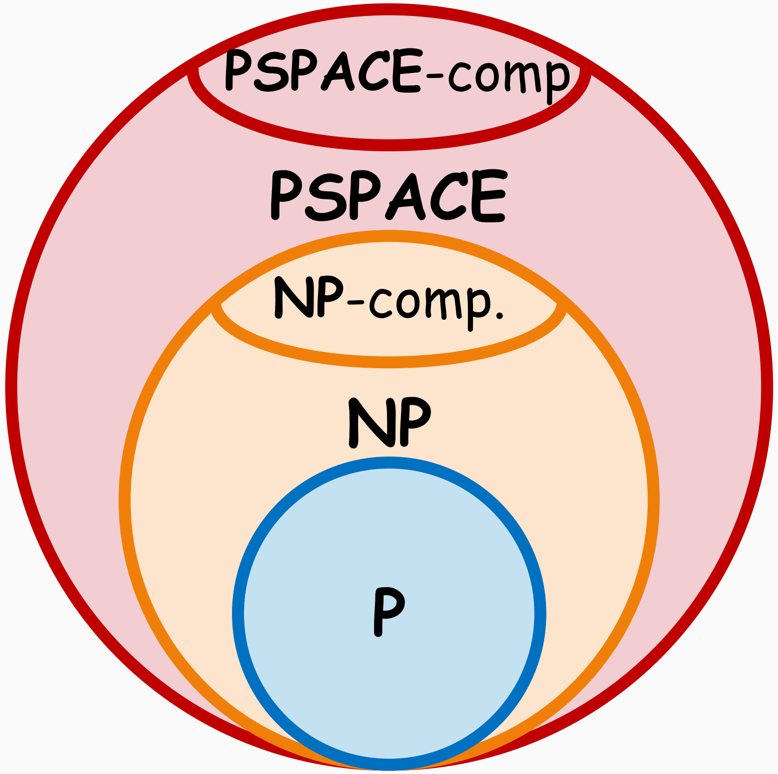
What we already know

- Maxmin 3-SAT Reconfiguration is…

~~P~~

~~NP-comp.~~

PSPACE-comp. [Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]



⚠ Approximate versions are (at least) harder than decision problems

Our focus

Three possible worlds(?)

- 0.999-approx. of Maxmin 3-SAT Reconfiguration is…

NP-comp.

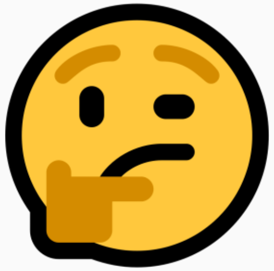
PSPACE-comp.

We only know **NP**-hardness (until recently) [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. ISAAC 2008 & Theor. Comput. Sci. 2011]

- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

So why we need **PSPACE**-completeness?

> It doesn't matter whether **NP**-hard or **PSPACE**-hard.



algorithm designer

- 1. **PSPACE**-completeness is tight
- 2. No efficient algorithm under **P** $\neq$ **PSPACE**
- 3. No short reconf. sequence under **NP** $\neq$ **PSPACE**

Significance of **PSPACE**-completeness

1. PSPACE-completeness is tight

🔁Reconfiguration problems of

Satisfiability, Independent Set, Coloring, Vertex Cover, Dominating Set, Clique,
Shortest Path, Hamiltonian Cycle, Feedback Vertex Set, Steiner Tree,
Vertex Separator, Odd Cycle Transversal, Induced Forest, L(2,1)-Labeling,
Integer Linear System, Target Set, Set Cover, Subset Sum, H-word

are **PSPACE**-complete

PSPACE-comp. of approx. implies...

😋Solving **approximately** is as hard as solving **exactly**

Significance of **PSPACE**-completeness

2. No efficient algorithm under P $\neq$ PSPACE

Proposition	RW's Estimated Likelihood
TRUE	100%
$\mathsf{EXP}^{\mathsf{NP}} \neq \mathsf{BPP}$	99%
$\mathsf{NEXP} \not\subset \mathsf{P/poly}$	97%
$\mathsf{L} \neq \mathsf{NP}$	95%
$\mathsf{NP} \not\subset \mathsf{SIZE}(n^k)$	93%
$\mathsf{BPP} \subseteq \mathsf{SUBEXP}$	90%
$\mathsf{P} \neq \mathsf{PSPACE}$	90%
$\mathsf{P} \neq \mathsf{NP}$	80%
ETH	70%

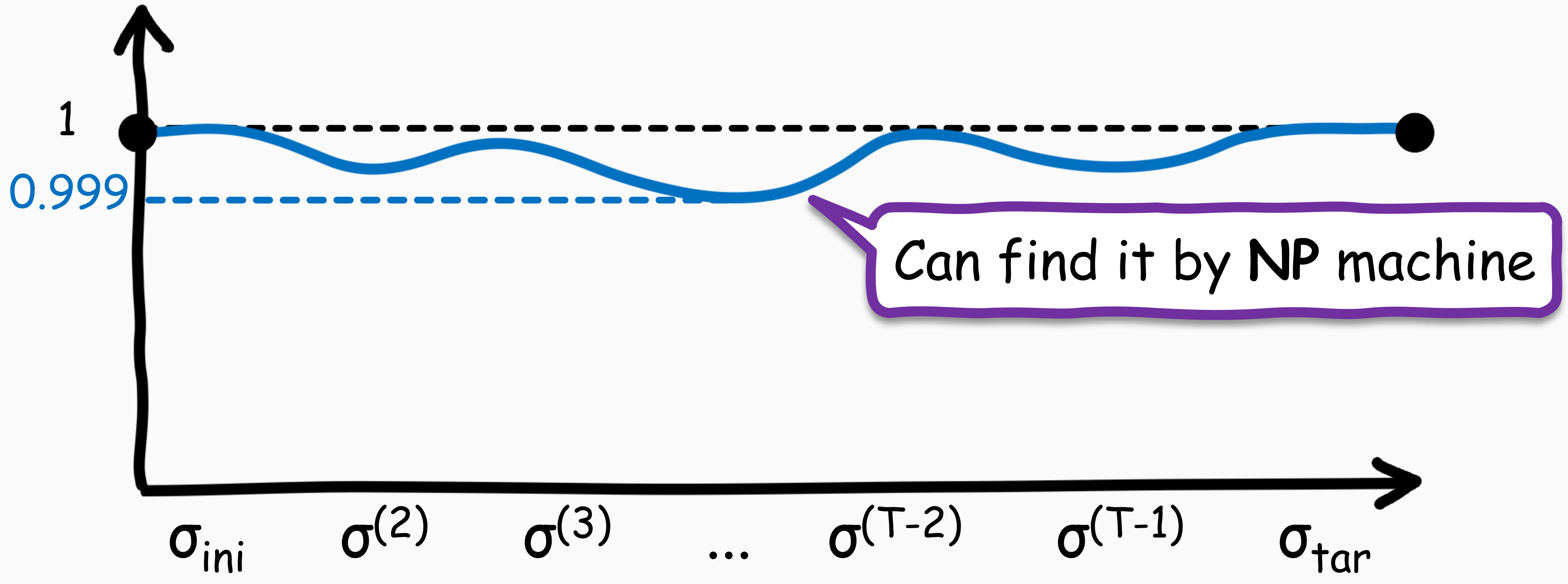
[Ryan Williams. "Some estimated likelihoods for computational complexity". 2019]

😋 Become **10%** more confident

Significance of **PSPACE**-completeness

3. No short reconf. seq. under **NP** ≠ **PSPACE**

Suppose "there is a 0.999-approx. reconf. sequence of **poly-length**"



⚠ Diameter of 3-SAT Reconf. can be $2^{\Omega(n)}$
[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

😋 Complexity results imply (some) **structural** properties

Formulating hardness of approximation

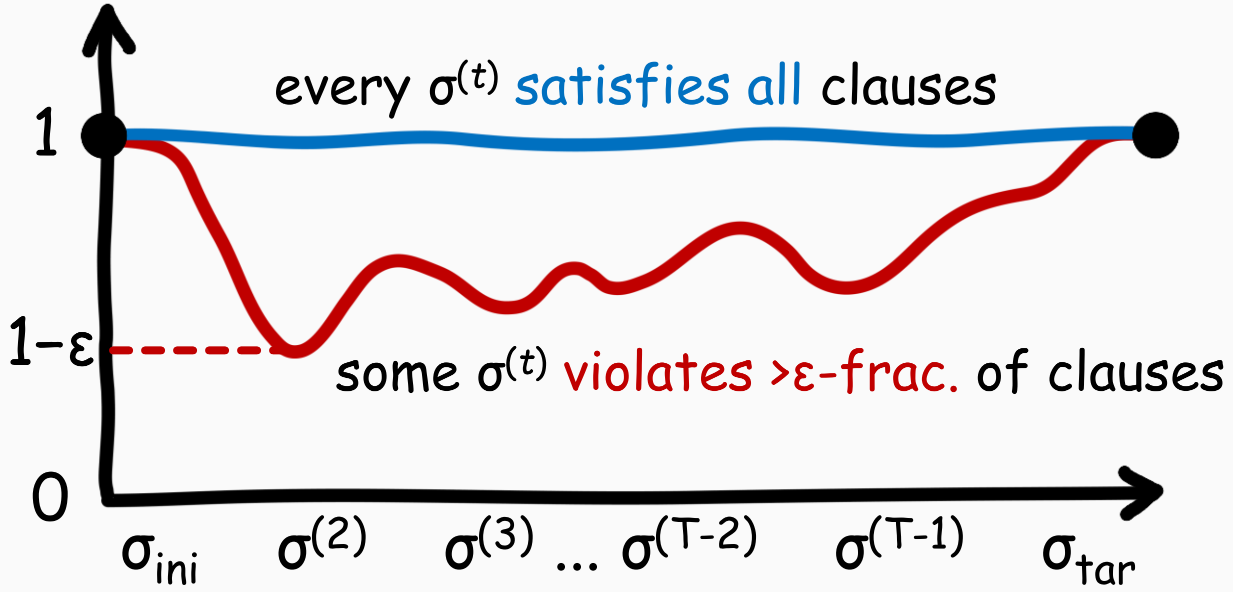
Gap[1 vs. 1−ε] 3-SAT Reconfiguration

- **Input**: φ & satisfying σ_{ini}, σ_{tar}
- **Goal**: Distinguish between

(Completeness) $\exists \vec{\sigma}$ $\mathrm{val}_\varphi(\vec{\sigma}) = 1$

(Soundness) $\forall \vec{\sigma}$ $\mathrm{val}_\varphi(\vec{\sigma}) < 1-\varepsilon$

$\mathrm{val}_\varphi(\vec{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)



Gap[1 vs. 1] 3-SAT Reconf. is **PSPACE**-comp.

Gap[1 vs. 0.5] 3-SAT Reconf. is **P**

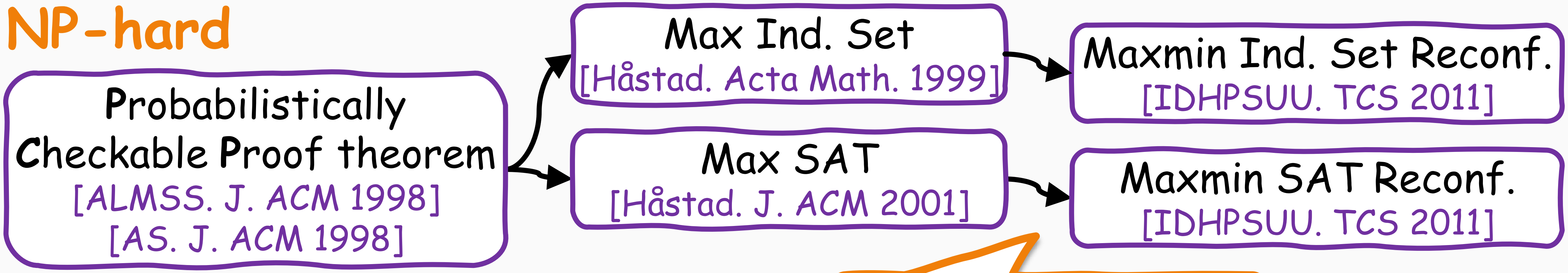
Gap[1 vs. 0.999] 3-SAT Reconf. is $\mathcal{C}$-hard

Studying gap problems is enough

$\Longrightarrow$ 0.999-approx. of Maxmin 3-SAT Reconf. is $\mathcal{C}$-hard

Known hardness-of-approx. results by 2022

NP-hard



What's going on?

PSPACE-complete

Exercise 3

Gap-preserving reduction from Max 3-SAT to Maxmin 5-SAT Reconf.

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Gap$_{[1 \text{ vs. } 1-\varepsilon]}$ 3-SAT φ

n variables : $x_1, \ldots, x_n$

m clauses : $C_1, \ldots, C_m$

→

Gap$_{[1 \text{ vs. } 1-\frac{\varepsilon}{2}]}$ 5-SAT Reconf. $(\psi, \sigma_{ini}, \sigma_{tar})$

n+2 vars. : $x_1, \ldots, x_n, y, z$

2m clauses : $C_1 \vee y \vee \bar{z}, \ldots, C_m \vee y \vee \bar{z}$
$C_1 \vee \bar{y} \vee z, \ldots, C_m \vee \bar{y} \vee z$

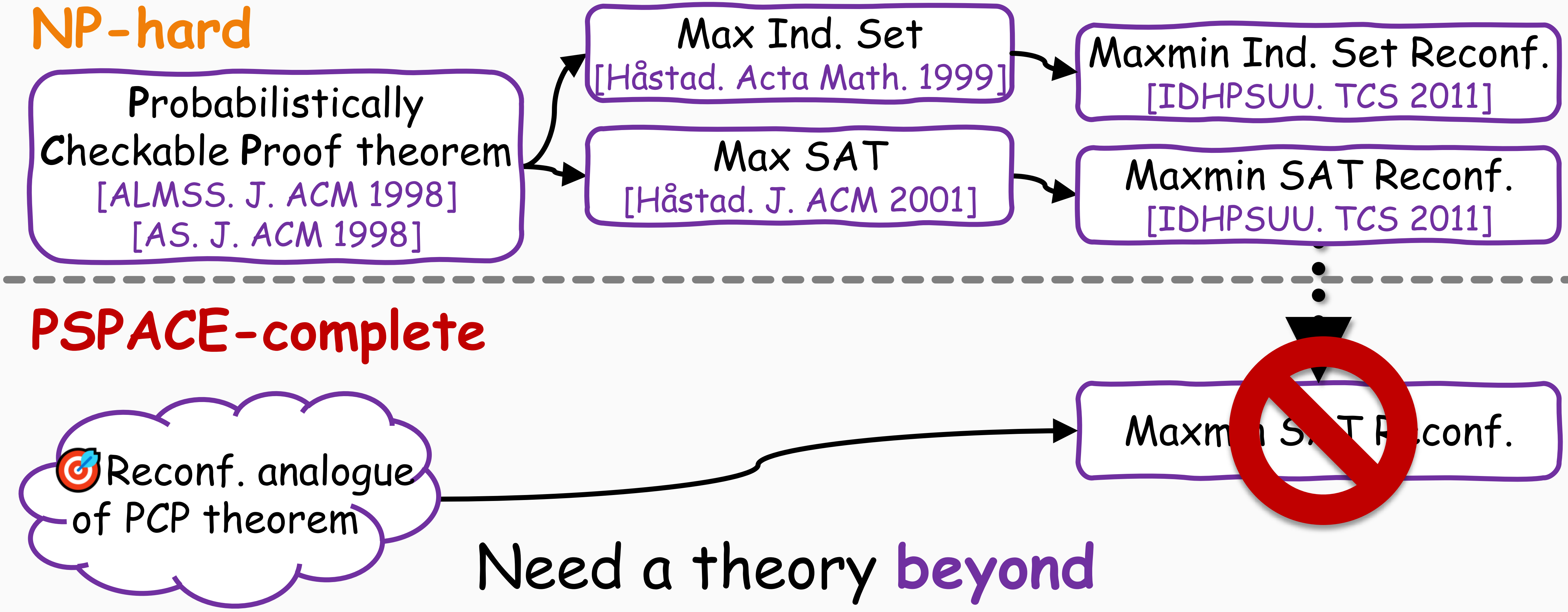
asgmnts. : $\sigma_{ini} := 0^{n+2}$ & $\sigma_{tar} := 1^{n+2}$

(Completeness) $\exists\sigma$ satisfies all clauses of $\varphi \implies \exists\vec{\sigma}\ \mathrm{val}_\psi(\vec{\sigma}) = 1$

(Soundness) $\forall\sigma$ violates >ε-frac. clauses of $\varphi \implies \forall\vec{\sigma}\ \mathrm{val}_\psi(\vec{\sigma}) < 1-\frac{\varepsilon}{2}$

⚠ $\vec{\sigma}$ must "touch" $y \neq z$ ➜ Half of clauses look like: $C_1 \wedge C_2 \wedge \cdots \wedge C_m$

Toward **PSPACE**-comp. of approx...



Need a theory **beyond** the PCP theorem for **NP**

Reconf. analogue of the PCP theorem circa 2023

[O. STACS 2023]

Reconfiguration Inapproximability Hypothesis

(formal statement omitted)

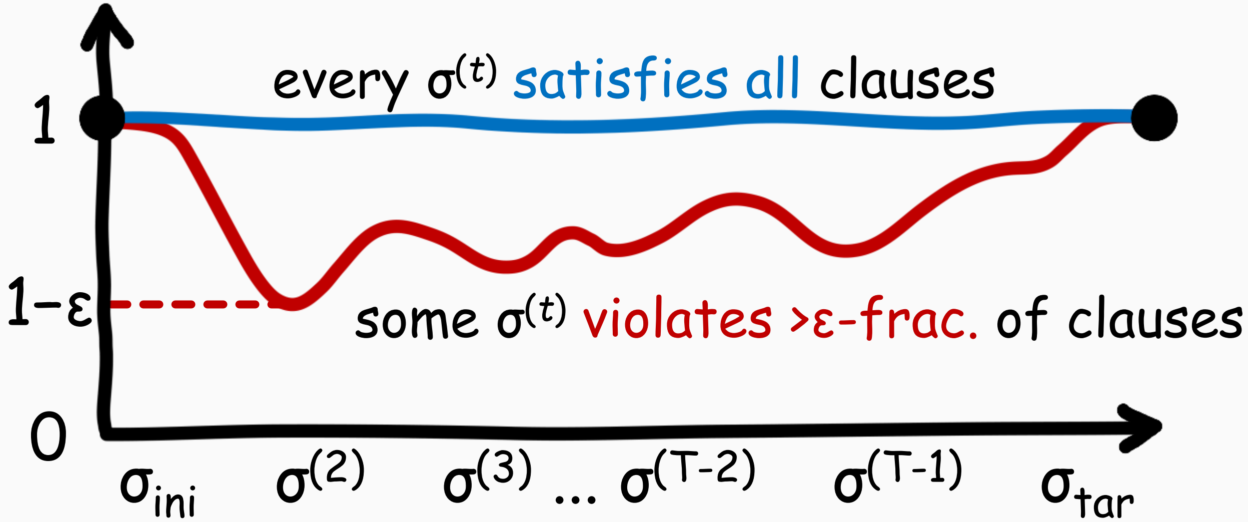
⇕ equivalent

"$\exists \varepsilon > 0$ Gap[1 vs. 1−ε] 3-SAT Reconf. is **PSPACE**-complete"

Goal: Distinguish between

(Completeness) $\exists \vec{\sigma}$ $\mathrm{val}_\varphi(\vec{\sigma}) = 1$

(Soundness) $\forall \vec{\sigma}$ $\mathrm{val}_\varphi(\vec{\sigma}) < 1-\varepsilon$



Settling the open problem conditional on RIH

NP-hard

Probabilistically Checkable Proof theorem
[ALMSS. J. ACM 1998]
[AS. J. ACM 1998]

Max Ind. Set
[Håstad. Acta Math. 1999]

Max SAT
[Håstad. J. ACM 2001]

Maxmin Ind. Set Reconf.
[IDHPSUU. TCS 2011]

Maxmin SAT Reconf.
[IDHPSUU. TCS 2011]

PSPACE-complete

Reconfiguration Inapproximability Hypothesis
[O. STACS 2023]

⚠ Is this true?

Maxmin Nondeterministic Constraint Logic

Maxmin 3-SAT Reconf.

Maxmin 2-SAT Reconf.

Maxmin Ind. Set Reconf.

Maxmin Clique Reconf.

Reuse [Garey-Johnson-Stockmeyer. Theor. Comput. Sci. 1976]

Cover Reconf.

Our main result

[Hirahara-O. STOC 2024] [Karthik C. S.-Manurangsi. 2023]

Probabilistically Checkable Reconfiguration Proof theorem

$\stackrel{\text{def}}{=}$ PCP-like characterization of **PSPACE**

For any language L in **PSPACE**

$\exists$ a verifier $\mathcal{V}$ with $O(\log n)$ randomness & $O(1)$ query complexity

$\exists$ poly-time alg. π_{ini} & π_{tar} s.t. for every input $x \in \{0,1\}^*$

$x \in L \implies \exists\ (\pi^{(1)}=\pi_{ini}(x), \ldots, \pi^{(T)}=\pi_{tar}(x))\quad \forall t\ \ \mathbf{Pr}[\mathcal{V}(x) \text{ accepts } \pi^{(t)}] = 1$

$x \notin L \implies \forall\ (\pi^{(1)}=\pi_{ini}(x), \ldots, \pi^{(T)}=\pi_{tar}(x))\quad \exists t\ \ \mathbf{Pr}[\mathcal{V}(x) \text{ accepts } \pi^{(t)}] < \frac{1}{2}$

The open problem resolved unconditionally

[Hirahara-O. STOC 2024] [Karthik C. S.-Manurangsi. 2023]

Probabilistically Checkable Reconfiguration Proof theorem

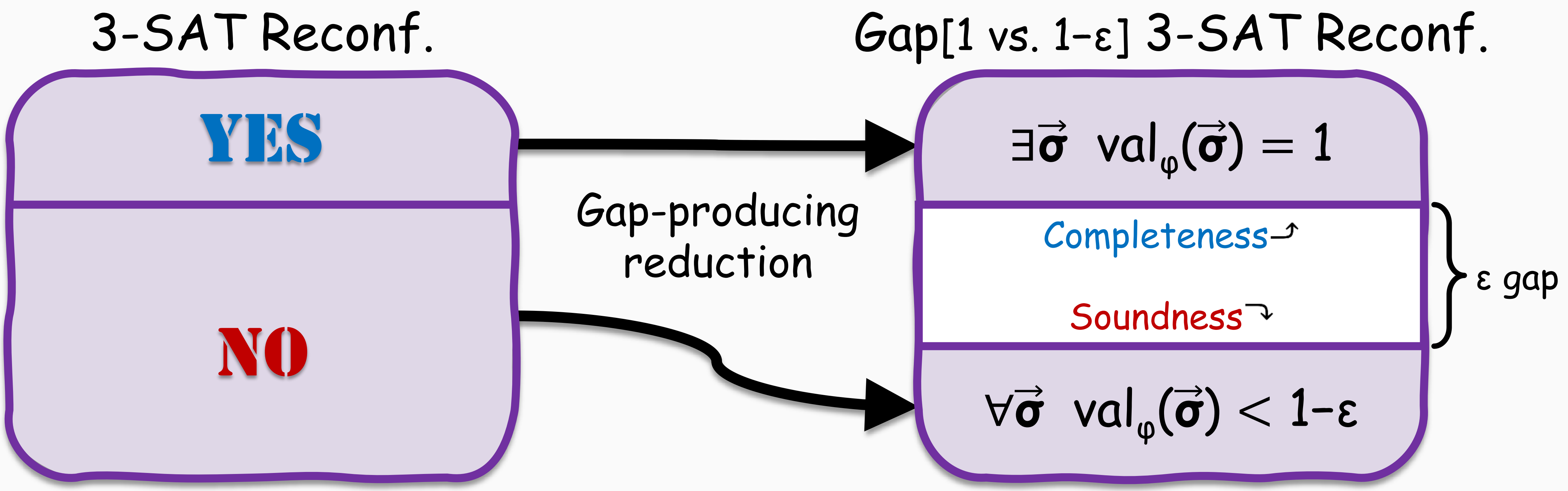
$\Updownarrow$ equivalent

$\exists \varepsilon > 0$ Gap[1 vs. 1−ε] 3-SAT Reconf. is **PSPACE**-complete

- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

How to prove the PCRP theorem?

[Hirahara-O. STOC 2024] [Karthik C. S.-Manurangsi. 2023]



- Our luck: **PCP of proximity** (a.k.a. assignment testers)
  [Ben-Sasson, Goldreich, Harsha, Sudan, Vadhan. SIAM J. Comput. 2006]
  [Dinur-Reingold. SIAM J. Comput. 2006]

So the story ends...?

Optimal **PSPACE**-completeness of approx.

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Recent progress, Struggles & future directions

Current status of Maxmin k-SAT Reconf.

- 1: **PSPACE**-comp. [Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]
- $1 - \Omega(2^{-k})$: **PSPACE**-comp. (PCRP thm.) [Hirahara-O. STOC 2024]
- $1 - \Omega(9^{-\sqrt{k}})$: **PSPACE**-comp. [Hirahara-O. 2024] + PCRP thm.
- ?: Q. $(1 - k^{-2})$-apx. in NP?
- $1 - \frac{1}{8k}$: **NP**-hard [Hirahara-O. 2024]
- $1 - \frac{2.5}{k}$: **P** [Hirahara-O. 2024]
- 0.5: **P** [exercise]

😭Gap

Review of recent progress

problem	polynomial time	PSPACE-complete
Maxmin k-SAT Reconf.	$1 - \frac{2.5}{k}$ [Hirahara-O. 2024]	$1 - \Omega(9^{-\sqrt{k}})$ [Hirahara-O. 2024]
Minmax Set Cover Reconf.	2 [IDHPSUU. Theor. Comput. Sci. 2011]	$2 - o(1)$ [Hirahara-O. ICALP 2024]
Maxmin Ind. Set Reconf.	n^{-1}	$n^{-0.001}$ [Hirahara-O. STOC 2024]
Maxmin 2-CSP Reconf.	0.5 [Karthik C. S.-Manurangsi. 2023]	0.9942 [O. SODA 2024]
Maxmin k-Cut Reconf.	$1 - \frac{2}{k}$ [Hirahara-O. 2024]	$1 - \Omega(\frac{1}{k})$ [Hirahara-O. 2024]

Struggles to transfer PCP tools to the reconfiguration world

existing PCP tools	techniques in reconf. world
FGLSS reduction [Feige-Goldwasser-Lovász-Safra-Szegedy. J. ACM 1996]	Alphabet squaring [Hirahara-O. ICALP 2024]
Degree reduction [Papadimitriou-Yannakakis. J. Comput. Syst. Sci. 1991]	Alphabet squaring [O. STACS 2023]
Gap amplification [Dinur. J. ACM 2007]	Alphabet squaring [O. SODA 2024]
Alphabet reduction [Dinur. J. ACM 2007]	Reconfigurability of Hadamard codes [O. ICALP 2024]
Parallel repetition theorem [Raz. SIAM J. Comput. 1998]	⚠ Applied to Max k-SAT [Håstad. J. ACM 2001]
Long code test [Bellare-Goldreich-Sudan. SIAM J. Comput. 1998]	

Some future directions

1. Source problems in P

- Shortest Path Reconf. is **PSPACE**-complete [Bonsma. Theor. Comput. Sci. 2013]
- **PSPACE**-complete to approximate as well?

2. Puzzles

- Study approximability of Sliding Block Puzzle (?) [Hearn-Demaine. Theor. Comput. Sci. 2005]
- 🎯Hardness of approx. for **planar** Nondeterministic Constraint Logic

3. Parameterized inapproximability

Logspace analogue of **XP**

- *k*-Clique Reconf. & *k*-Dominating Set Reconf. are "**XL**-complete" [Bodlaender-Groenland-Nederlof-Swennenhuis. FOCS 2021] [Bodlaender-Groenland-Swennenhuis. IPEC 2021]
- **XL**-complete to approximate?

Conclusion & Takeaway

Resolved 4th open problem of
[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Now ready to study

hardness of approx. & approx. algorithms

for reconfiguration problems

THANK YOU!