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Asymptotically Optimal Inapproximability of Ek-SAT RECONFIGURATION

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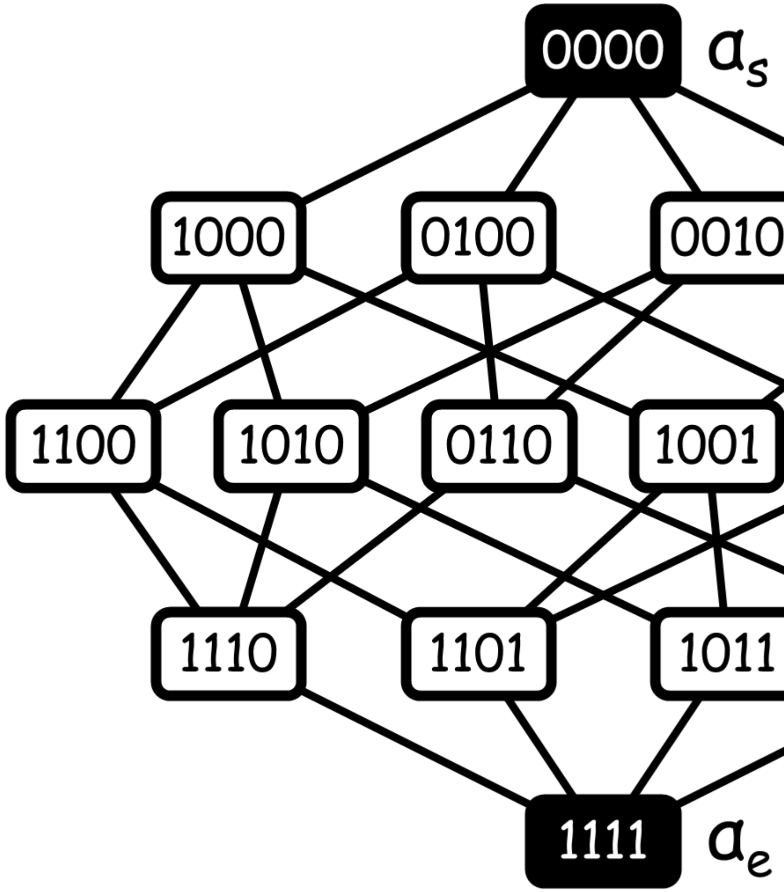
Example 1

E3-SAT RECONFIGURATION

[Gopalan-Kolaitis-Maneva-Papadimitriou 2009]

3-CNF formula $\varphi = (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4})$

Q. Path of satisfying assignments from a_s to a_e?



👻 Example 1

E3-Sat Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou 2009]

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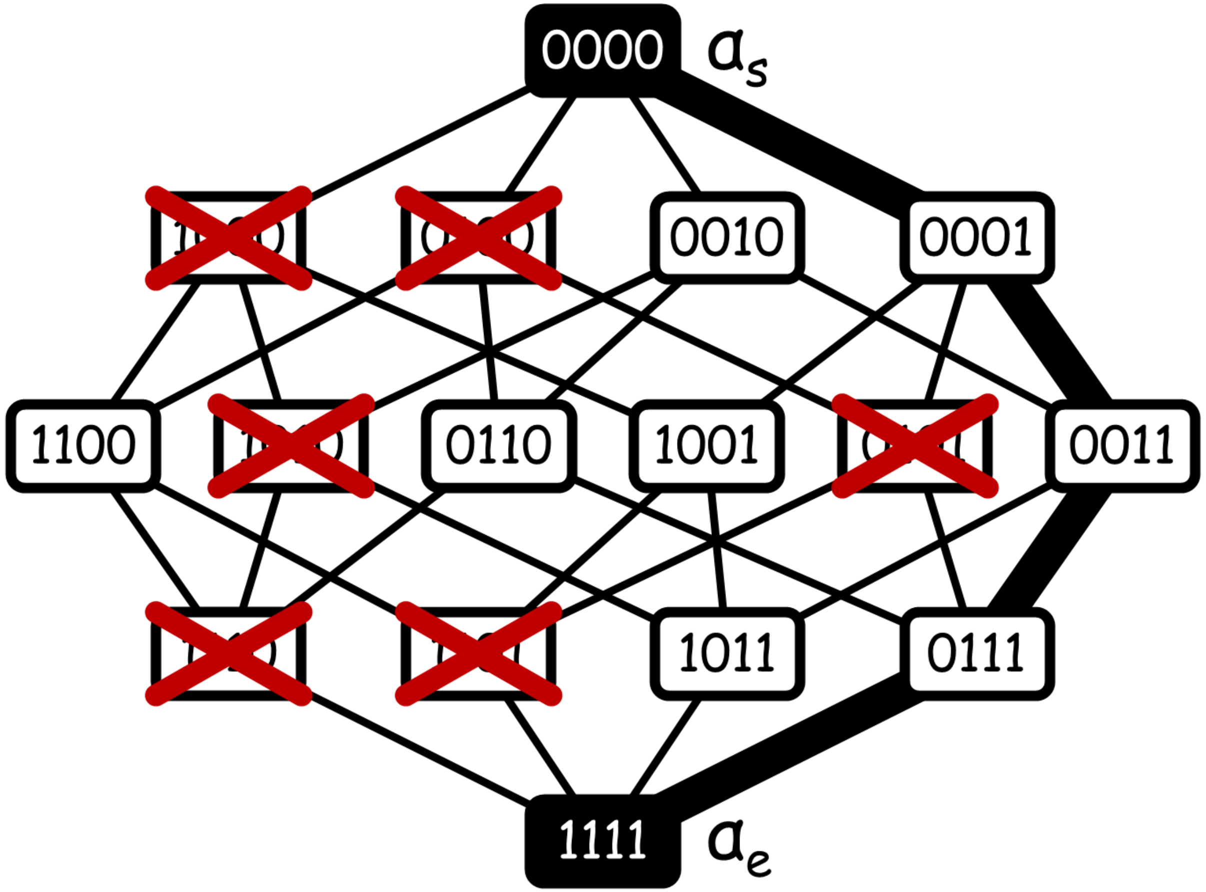

YES

Example 2

E3-Sat Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou 2009]

3-CNF formula $\varphi = (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4})$

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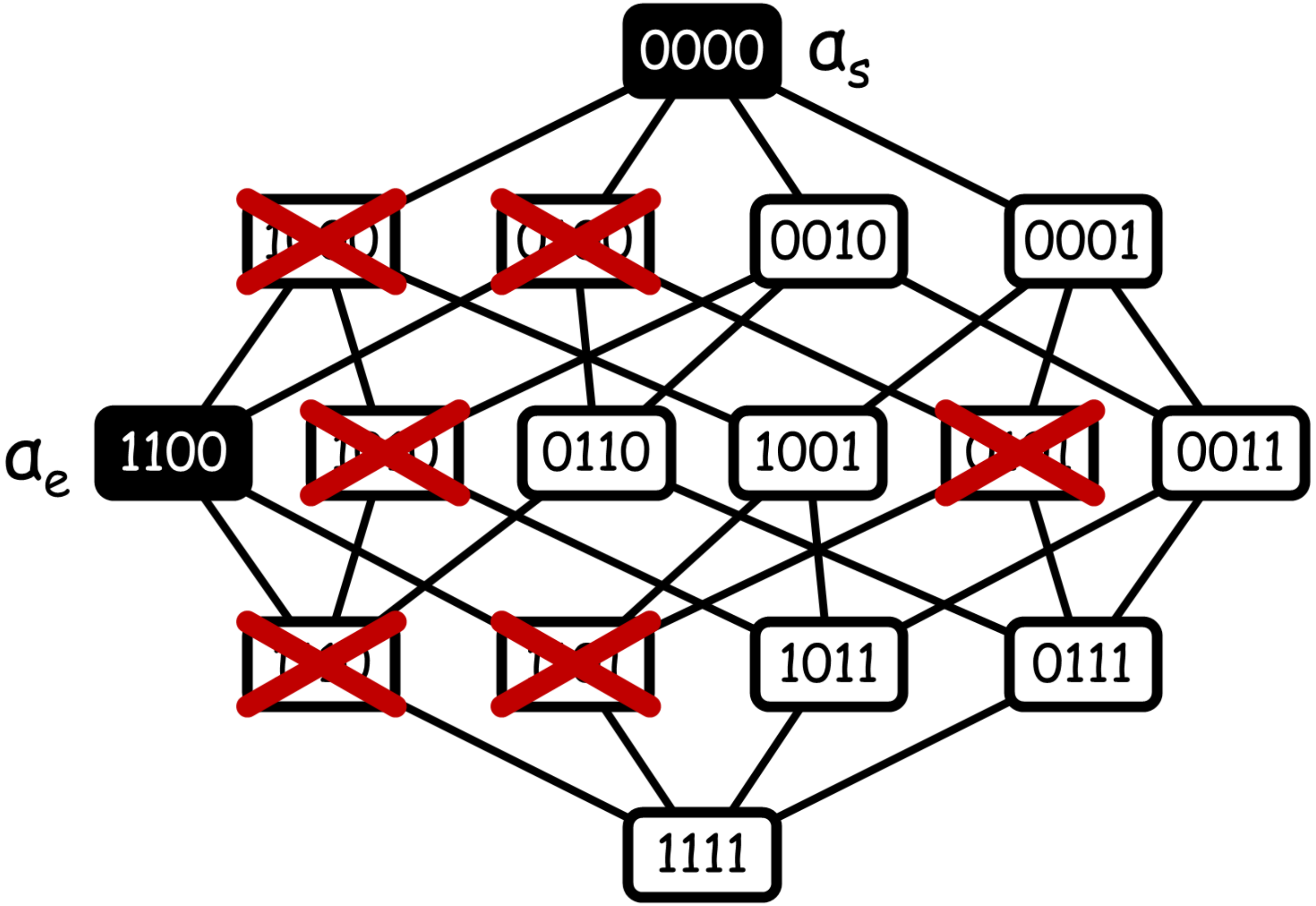

Example 2

E3-Sat Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou 2009]

3-CNF formula $\varphi = (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4})$

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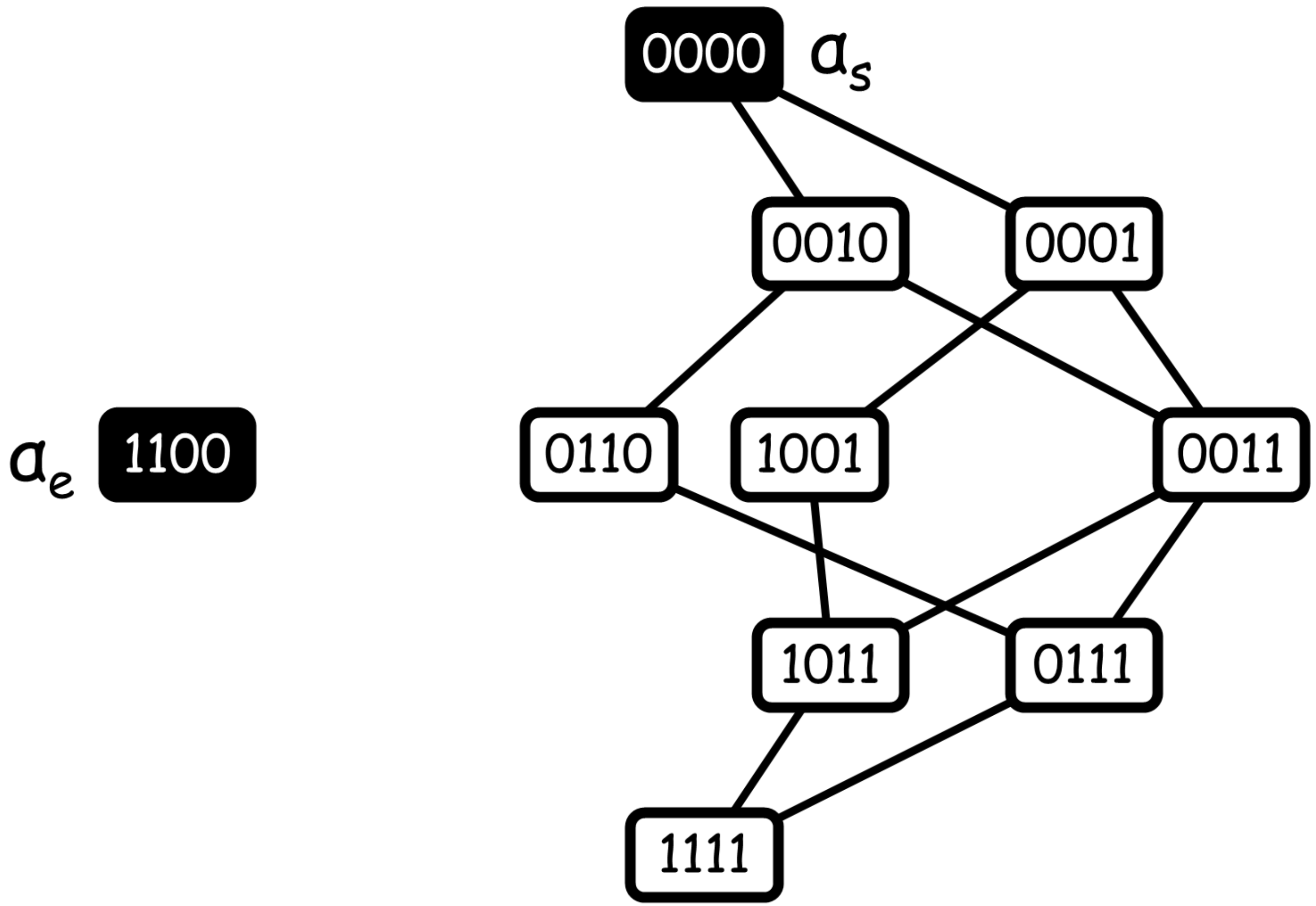


NO

Motivation

Solution space for Boolean formulas

For random instances

- Solution space breaks down into exponentially many "clusters"
  [Achlioptas/Coja-Oghlan/Ricci-Tersenghi 2011] [Mézard–Mora–Zecchina 2005]
- 👆 Explain SAT solver performance
  DPLL [Achlioptas–Beame–Molloy 2004] & Survey Propagation [Mézard–Parisi–Zecchina 2002]

In the worst-case scenario

- Dichotomy theorem [Gopalan–Kolaitis–Maneva–Papadimitriou 2009]
- 😋 Every reconfiguration problem over Boolean formulas is

P or **PSPACE-complete**

(linear diameter) (exponential diameter)

Example 3

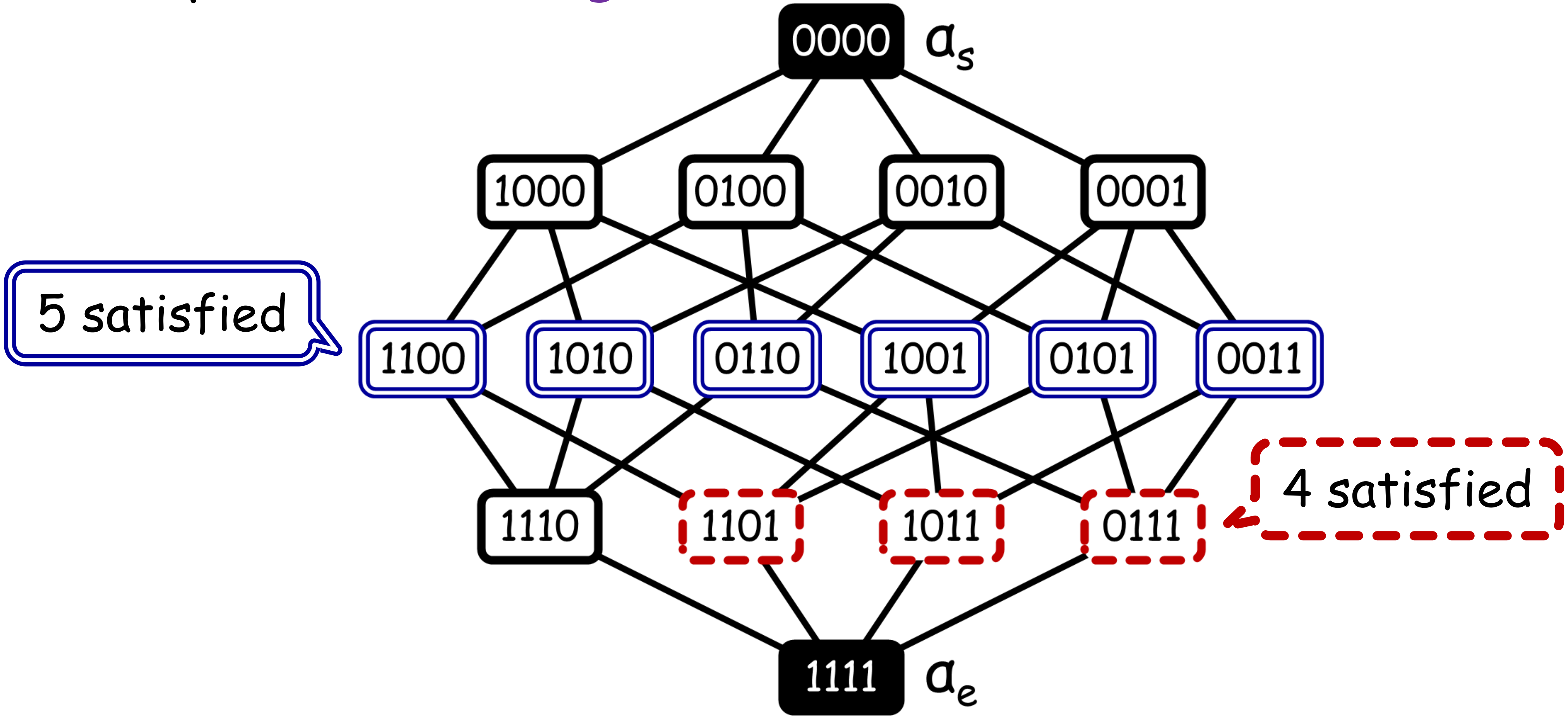
Maxmin E3-Sat Reconfiguration

approximate version

[Ito–Demaine–Harvey–Papadimitriou–Sideri–Uehara–Uno 2011]

$$\psi = (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_4})$$

Q. Find a path maximizing the minimum fraction of satisfied clauses



Example 3

MAXMIN E3-SAT RECONFIGURATION

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno 2011]

$$\psi = (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_4})$$

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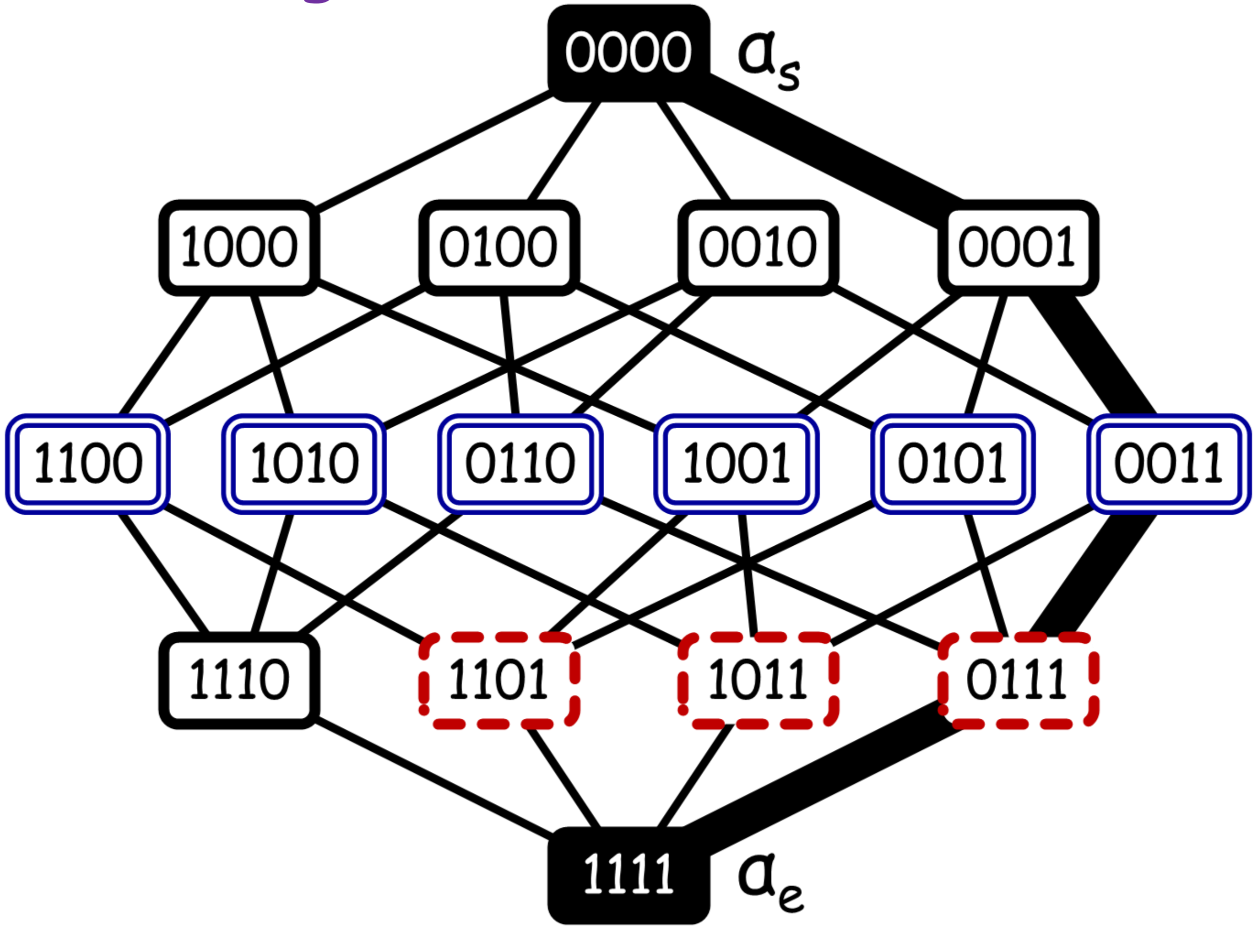

$$\frac{4}{6}$$

🫣 Example 3

Maxmin E3-Sat Reconfiguration

[Ito–Demaine–Harvey–Papadimitriou–Sideri–Uehara–Uno 2011]

$$\psi = (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_4})$$

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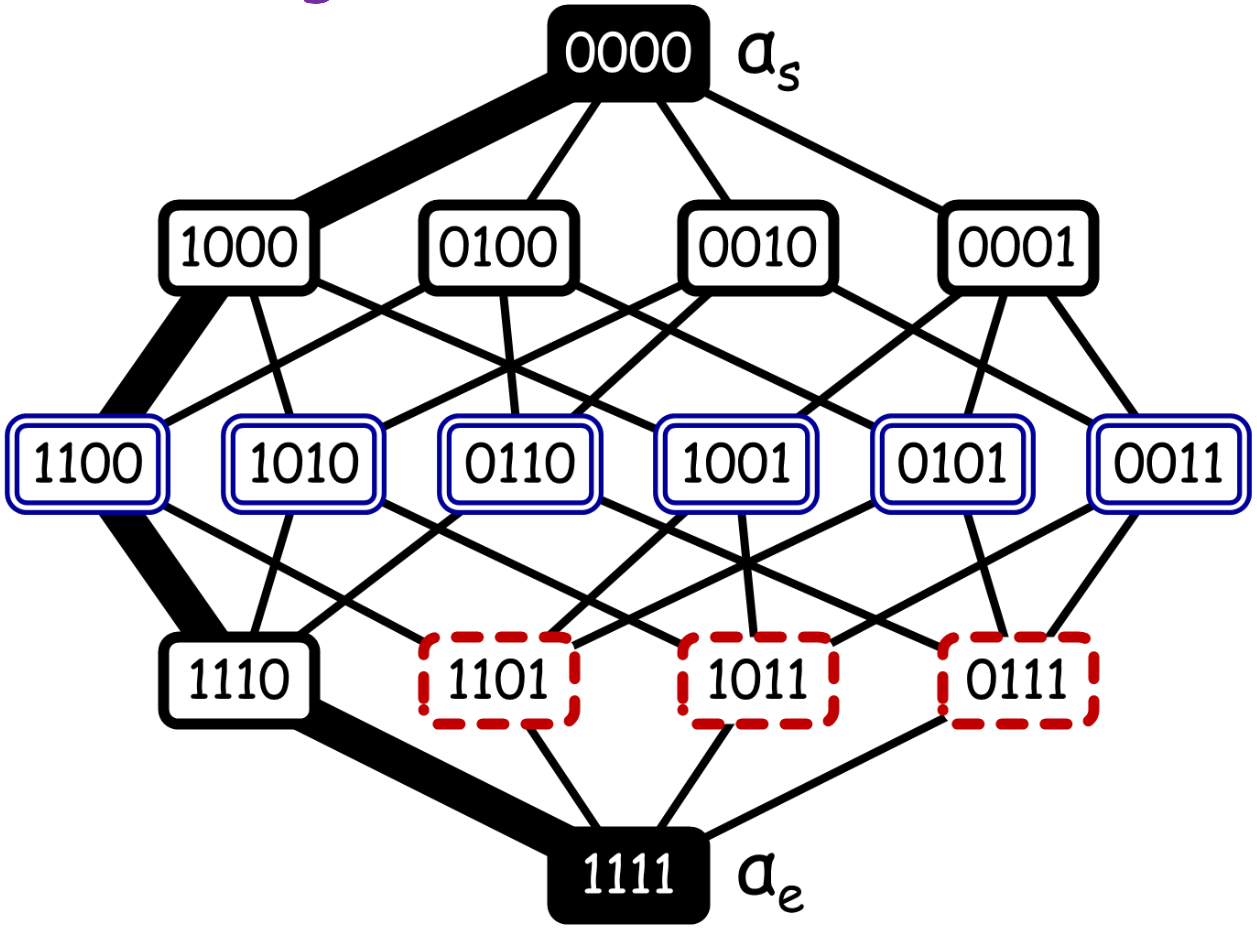


$$\frac{5}{6}$$

Complexity of Maxmin Ek-SAT Reconf

- **PSPACE**-hard to solve exactly $\forall k \geq 3$
  [Gopalan-Kolaitis-Maneva-Papadimitriou 2009]

- **NP**-hard to approximate within $\frac{15}{16}+\varepsilon$ if $k = 5$
  [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno 2011]

- **PSPACE**-hard to approximate if $k = 3$
  [Hirahara-O. STOC 2024] [Karthik C. S.-Manurangsi 2023] [O. STACS 2023]

Q. What is **asymptotic behavior** of approximability w.r.t. the clause width k?

Our results

Optimal approximation factor $= 1 - \Theta\left(\frac{1}{k}\right)$

PSPACE-complete — $1 - \frac{1}{10k}$ (for all sufficiently large k)

NP-hard — $1 - \frac{1}{8k}$

P — $1 - \frac{2.5}{k}$

So what?

Similar results are known for reconf problems
[Hirahara-O. STOC 2024] [Hirahara-O. ICALP 2024 & 2025]
[Karthik C. S.–Manurangsi 2023] [O. STACS 2023]
[O. SODA 2024] [O. ICALP 2024 & 2025]

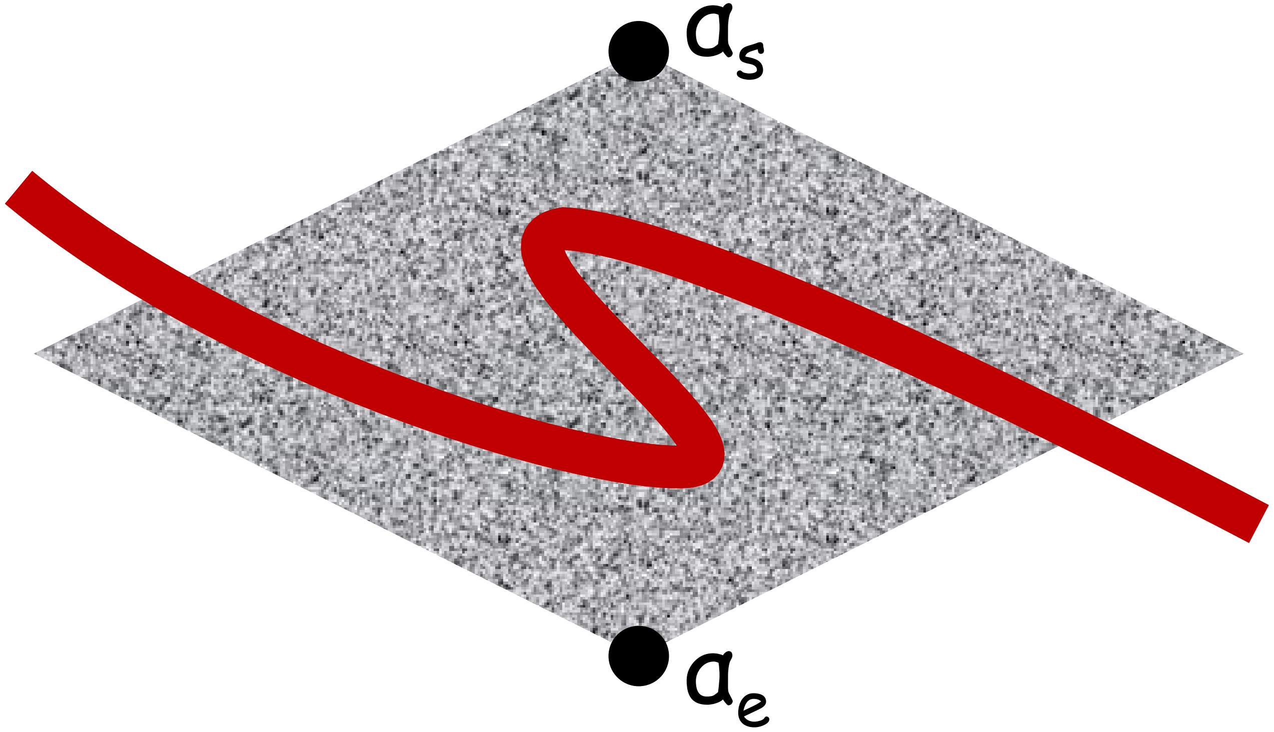
Why surprising (at least to me)?

Implications for solution-space structure

🔁 Random asgmt satisfies $(1 - \frac{1}{2^k})$-frac of clauses (in expectation)

∴ Only $2^{-\Omega(n)}$-frac of asgmts do not satisfy $(1 - \frac{1}{10k})$-frac of clauses

∴ Deleting "rare" bad asgmts makes *st*-CONNECTIVITY **PSPACE**-hard



Why surprising (at least to me)?

First reconfiguration problem that is harder to approximate than its NP analogue

problem	threshold	hardness	refs.
MAXMIN Ek-SAT RECONF	$1 - \Theta(k^{-1})$	$\boxed{\text{Harder}}$ CE-h	(this work)
MAX Ek-SAT	$1 - 2^{-k}$	NP-h	[Håstad 2001]
MAXMIN k-CUT RECONF	$1 - \Theta(k^{-1})$	$\boxed{\text{Equal}}$ CE-h	[Hirahara-O. 2025]
MAX k-CUT	$1 - \Theta(k^{-1})$	NP-h	[Frieze-Jerrum 1997] [Kann-Khanna-Lagergren-Panconesi 1997]
MAXMIN 2-CSP RECONF	$\frac{1}{2}$	$\boxed{\text{Easier}}$ CE-h	[Karthik C. S.-Manurangsi 2023] [O. 2024 & 2025]
MAX 2-CSP	$(\text{poly } N)^{-1}$	NP-h	[Charikar-Hajiaghayi-Karloff 2011] [Raz 1998]
MINMAX SET COVER RECONF	2	$\boxed{\text{Easier}}$ CE-h	[Hirahara-O. 2024] [IDHPSUU 2011] [Karthik C. S.-Manurangsi 2023]
MIN SET COVER	$\ln N$	NP-h	[Chvátal 1979] [Feige 1998] [Johnson 1974] [Lovász 1975]

In the remainder of this talk...

Proof overview of **PSPACE**-hardness of $\left(1 - \Omega\left(\frac{1}{1.92^k}\right)\right)$-factor approximation

Still nontrivial

Proof overview

Gap[1, 1−ε] E3-SAT RECONFIGURATION

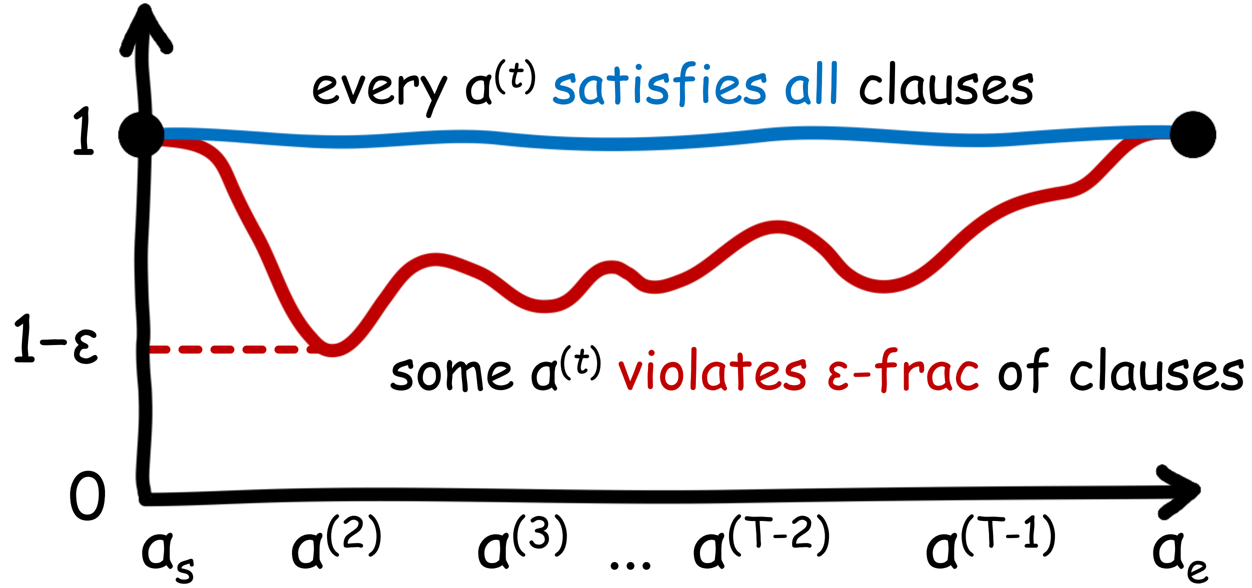
Given 3-CNF formula φ & satisfying assignments α_s, α_e

$\mathrm{opt}_\varphi(\alpha_s, \alpha_e) :=$ optimal value of MAXMIN E3-SAT RECONFIGURATION

🎯 Distinguish between

(Completeness) $\mathrm{opt}_\varphi(\alpha_s, \alpha_e) = 1$

(Soundness) $\mathrm{opt}_\varphi(\alpha_s, \alpha_e) < 1-\varepsilon$



Proof overview

$1 - \Omega(1.92^{-k})$ inapproximability

Consider 3 PCP verifiers $(k := 3\lambda)$

Gap (completeness) – (soundness)

(Step 0)	Gap$_{[1,\,1-\varepsilon]}$ E3-SAT RECONF	ε
(Step 1)	3λ-query Horn verifier V_{Horn}	$\Omega(\frac{\varepsilon}{k})$
(Step 2)	3λ-query OR verifier V_{OR}	$\Omega(\frac{\varepsilon}{1.92^k})$

described by k-CNF formula

🎯 Find a path **maximizing** the **minimum** acceptance probability of V

$\mathrm{opt}_V(\alpha_s, \alpha_e) :=$ optimal value of verifier V

Proof overview

(Step 1) 🔮 3λ-query Horn verifier V_{Horn}

- **Input:** 3-CNF formula $\varphi = C_1 \wedge \ldots \wedge C_m$
- **Oracle access:** assignment a

> sample $i_1, i_2, \ldots, i_\lambda \sim [m]$
> **If** $C_{i_1} \vee \overline{C_{i_2}} \vee \cdots \vee \overline{C_{i_\lambda}}$ is satisfied by a **then**
>   **return** 1
> **else**
>   **return** 0

Horn-like condition

(Completeness) $\mathrm{opt}_\varphi(a_s, a_e) = 1 \implies \mathrm{opt}_{Horn}(a_s, a_e) = 1$

(Soundness) $\mathrm{opt}_\varphi(a_s, a_e) < 1 - \varepsilon \implies \mathrm{opt}_{Horn}(a_s, a_e) < 1 - \Omega(\frac{\varepsilon}{\lambda})$

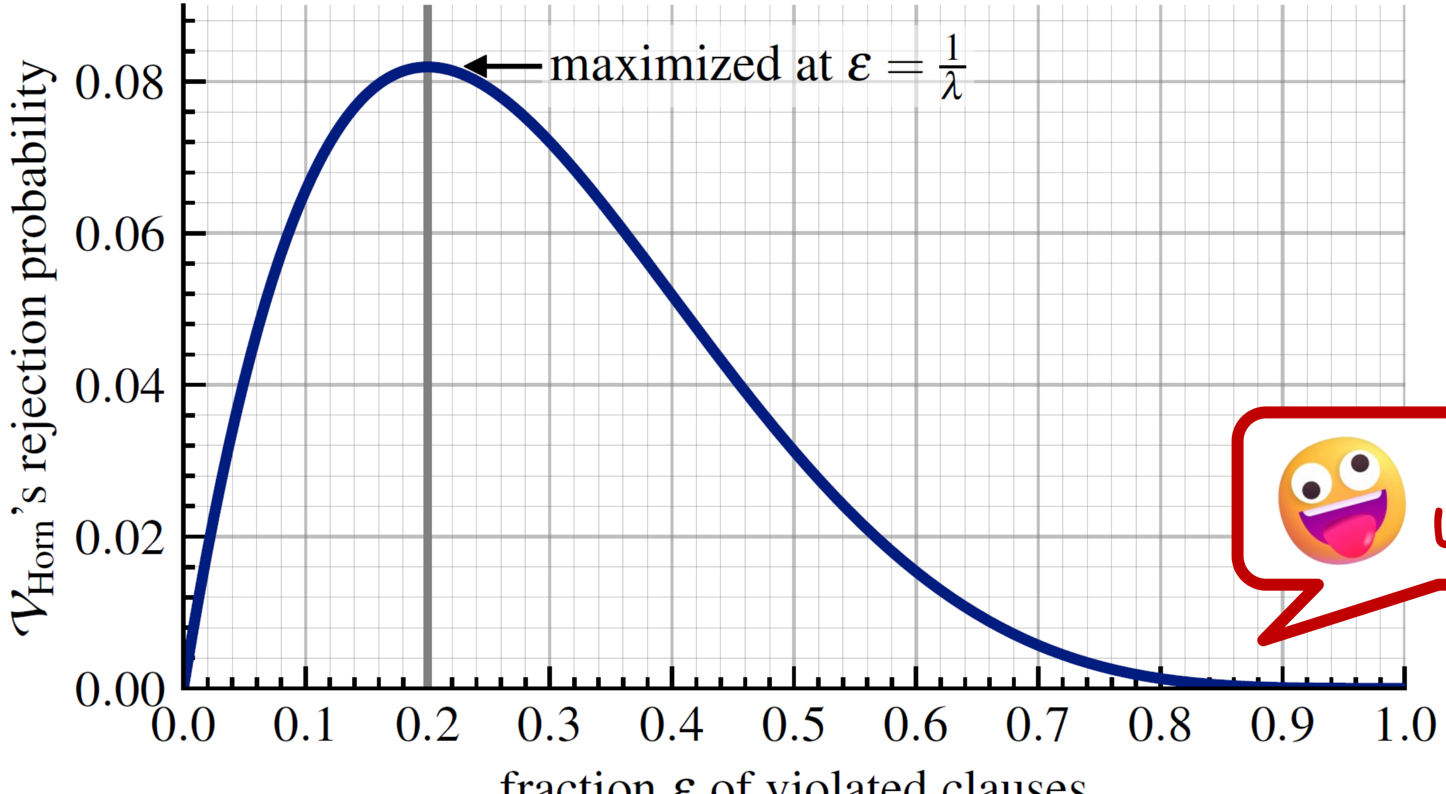
🤔 Why?

Proof overview

(Step 1) 🔑 V_{Horn} is "non-monotone"

$\varepsilon := \Pr_i[C_i \text{ is violated by } a] \quad \Longrightarrow \quad \Pr[V_{\text{Horn}} \text{ rejects } a] = \varepsilon(1-\varepsilon)^{\lambda-1}$

"$\overline{C_{i_1}} \wedge C_{i_2} \wedge \cdots \wedge C_{i_\lambda}$ is true"



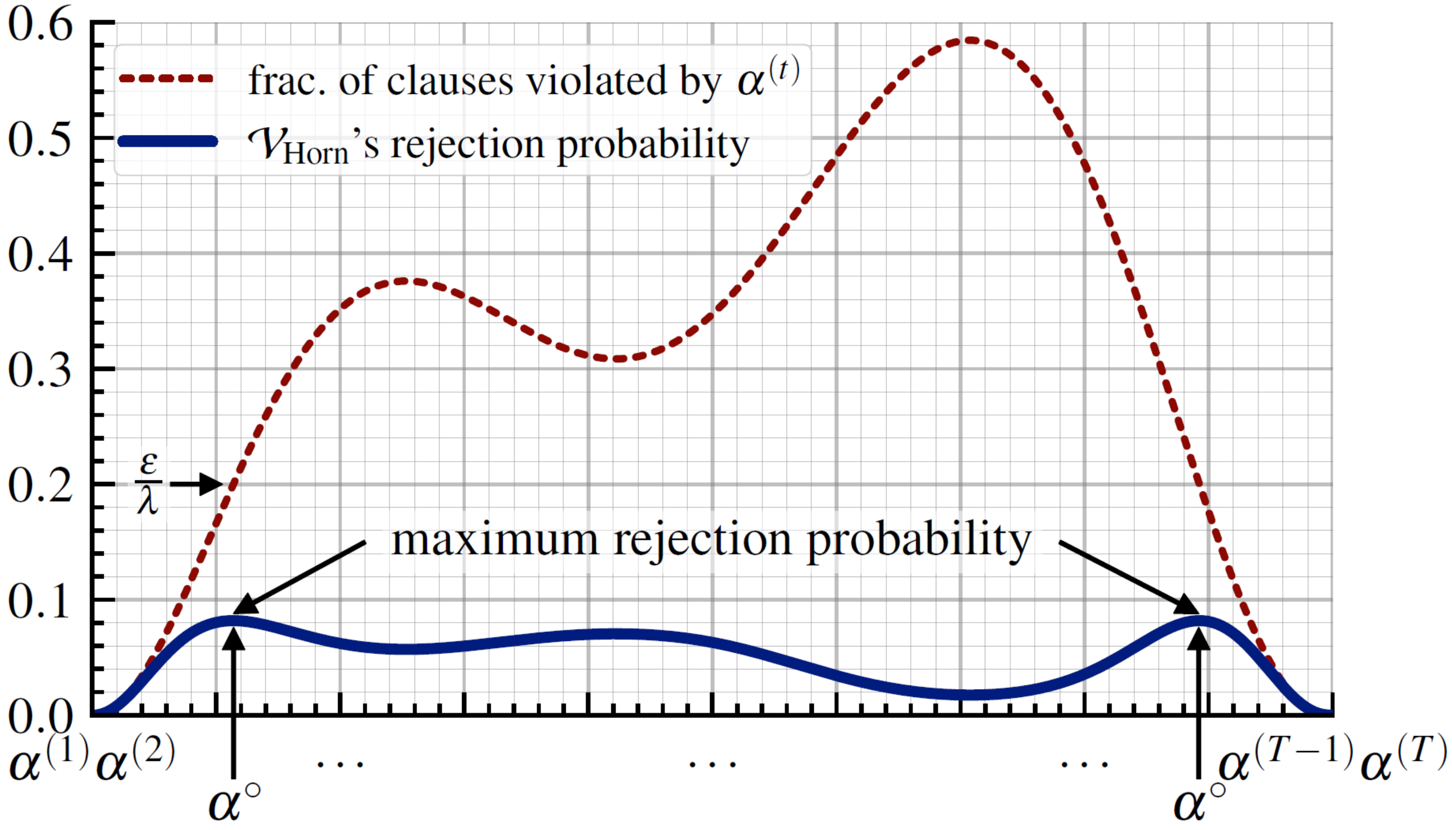
useless

Proof overview

(Step 1) 🪄 V_{Horn} is "useful"

💡 any path from α_s to α_e contains α° s.t. $\Pr_i[C_i \text{ is violated by } \alpha^\circ] \approx \frac{\varepsilon}{\lambda}$

$\Rightarrow$ 😋 $\Pr[V_{\text{Horn}} \text{ rejects } \alpha^\circ] \approx \frac{\varepsilon}{\lambda}(1 - \frac{\varepsilon}{\lambda})^{\lambda-1} = \Omega(\frac{\varepsilon}{\lambda})$



Proof overview

(Step 2) 3λ-query OR verifier V_{OR}

🔁 V_{Horn}'s rejection condition:

💡 (# rejecting local views) = $7^{\lambda-1}$

$$\overline{C_{i_1}} \wedge C_{i_2} \wedge \cdots \wedge C_{i_\lambda}$$

😋 Emulate V_{Horn} by an OR-predicate verifier V_{OR} s.t.

$$\mathbf{Pr}[V_{OR} \text{ rejects } a] = \mathbf{Pr}[V_{Horn} \text{ rejects } a] \cdot \frac{1}{7^{\lambda-1}}$$

(Completeness) $\mathrm{opt}_{Horn}(a_s, a_e) = 1 \implies \mathrm{opt}_{OR}(a_s, a_e) = 1$

(Soundness) $\mathrm{opt}_{Horn}(a_s, a_e) < 1 - \Omega(\frac{\varepsilon}{\lambda}) \implies$

$$\mathrm{opt}_{OR}(a_s, a_e) < 1 - \Omega\left(\frac{\varepsilon}{\lambda 7^{\lambda-1}}\right) = 1 - \Omega\left(\frac{\varepsilon}{1.92^k}\right)$$

Conclusions & open problems

🔁 Approximation threshold of MAXMIN Ek-SAT RECONF is 1 – $\Theta(\frac{1}{k})$

❓ Find more applications of "non-monotone" verifiers

❓ Find a class of Boolean relations for which MAXMIN SAT RECONF is harder to approximate than MAX SAT

THANK YOU!