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Alphabet Reduction for **Reconfiguration** Problems

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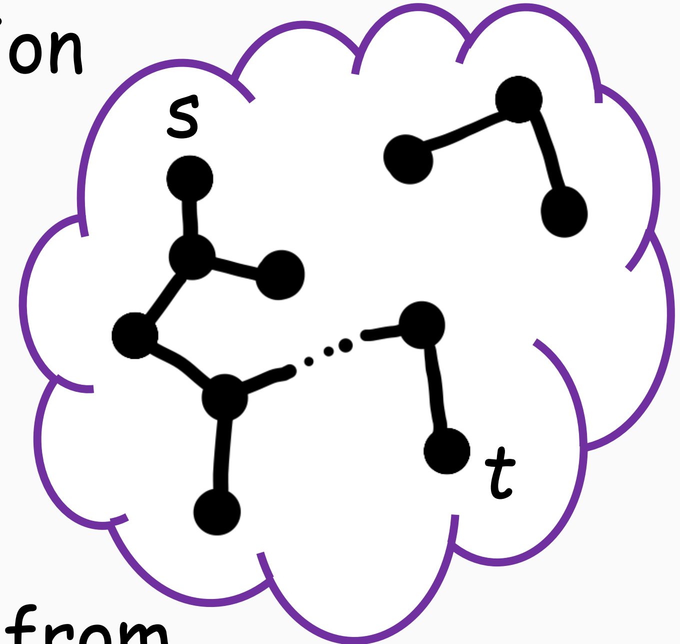
What is Combinatorial Reconfiguration...?

Imagine **connecting** a pair of feasible solutions (of NP problem)
under a particular adjacency relation



Q. Is a pair of solutions reachable to each other?
Q. If so, what is the shortest transformation?
Q. If not, how can the feasibility be relaxed?

Many reconfiguration problems have been derived from

Satisfiability, Coloring, Vertex Cover, Clique, Dominating Set, Feedback Vertex Set, Steiner Tree, Matching, Spanning Tree, Shortest Path, Set Cover, Subset Sum, ...

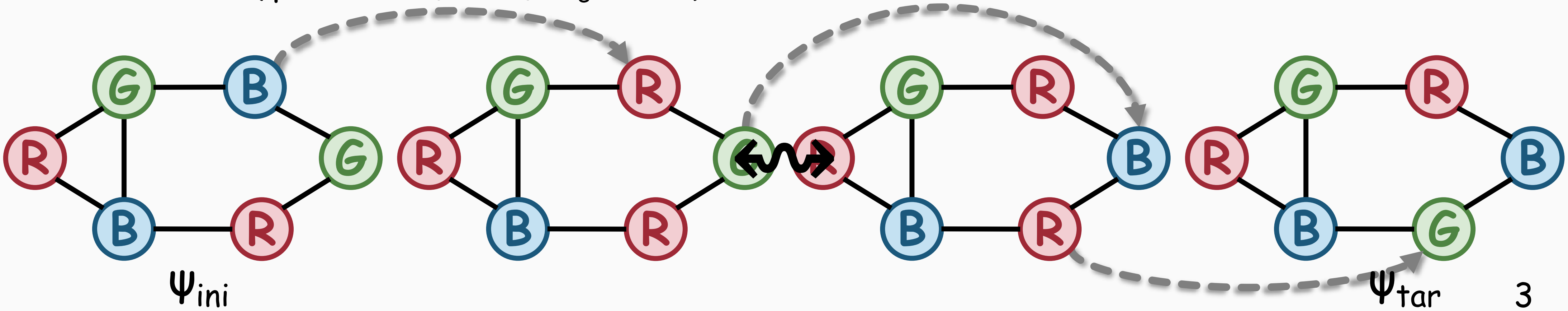
See [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011] [Nishimura. Algorithms 2018] [van den Heuvel. Surv. Comb. 2013] [Hoang. https://reconf.wikidot.com/]

Example 1

q-CSP Reconfiguration

- **Input:** q-ary CSP $G = (V,E,\Sigma,(\pi_e)_{e\in E})$ where $\pi_e: \Sigma^e \to \{0,1\}$
  satisfying $\psi_{ini}, \psi_{tar}: V \to \Sigma$
- **Output:** $\psi = (\psi^{(1)} := \psi_{ini}, \ldots, \psi^{(T)} := \psi_{tar})$ (reconf. sequence) s.t.
  every $\psi^{(t)}$ satisfies all edges of G (feasibility)
  $\mathrm{Ham}(\psi^{(t)}, \psi^{(t+1)}) \leq 1$ (adjacency)

YES case ($q = 2$, $\Sigma = \{R,G,B\}$, $\pi_e :=$ "$\neq$")

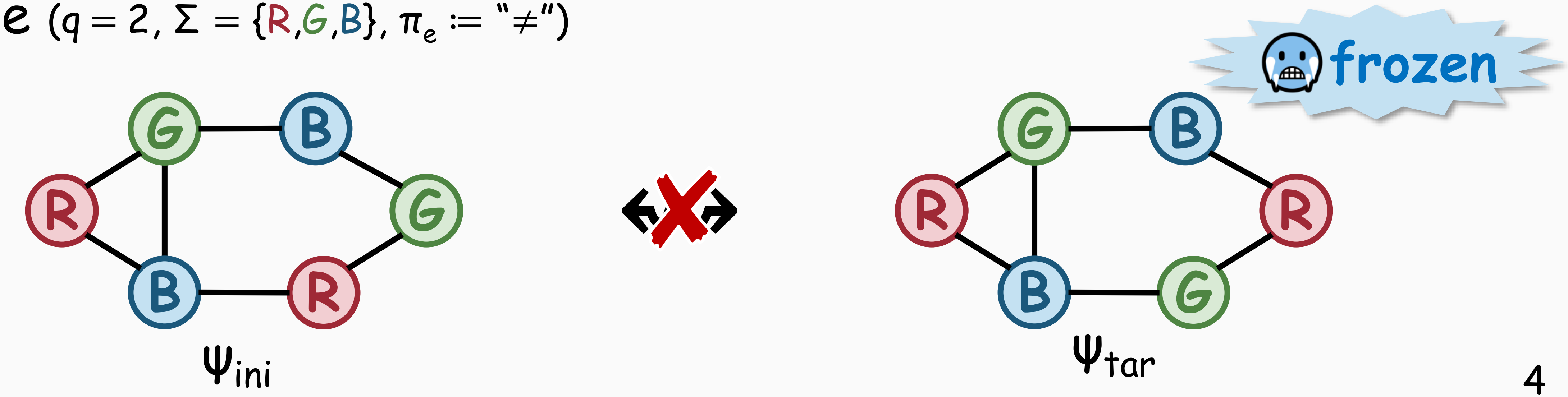


Example 2

q-CSP Reconfiguration

- **Input:** q-ary CSP $G = (V, E, \Sigma, (\pi_e)_{e \in E})$ where $\pi_e \colon \Sigma^e \to \{0,1\}$
  satisfying $\psi_{ini}, \psi_{tar} \colon V \to \Sigma$
- **Output:** $\psi = (\psi^{(1)} := \psi_{ini}, \ldots, \psi^{(T)} := \psi_{tar})$ (reconf. sequence) s.t.
  every $\psi^{(t)}$ satisfies all edges of G (feasibility)
  $\mathrm{Ham}(\psi^{(t)}, \psi^{(t+1)}) \leq 1$ (adjacency)

NO case ($q = 2$, $\Sigma = \{R, G, B\}$, $\pi_e :=$ "$\neq$")

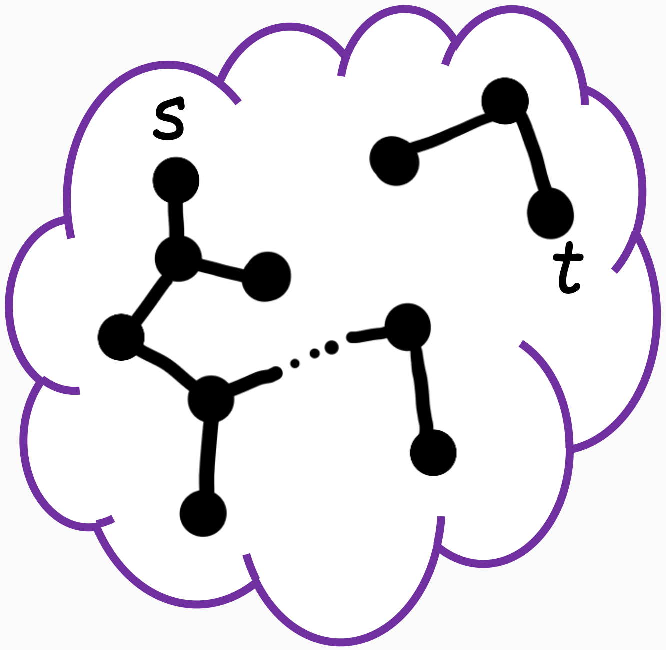


Complexity of reconfiguration problems

Source problem	Existence	Reconfiguration
Satisfiability	NP-complete	**PSPACE**-complete [Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]
Independent Set	NP-complete	**PSPACE**-complete [Hearn-Demaine. Theor. Comput. Sci. 2005]
Matching	P	P [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
3-Coloring	NP-complete	P [Cereceda-van den Heuvel-Johnson. J. Graph Theory 2011]
Shortest Path	P	**PSPACE**-complete [Bonsma. Theor. Comput. Sci. 2013]
Independent Set on bipartite graphs	P	NP-complete [Lokshtanov-Mouawad. ACM Trans. Algorithms 2019; SODA 2018]

👆 **Nontrivial relation**

Optimization versions of reconfiguration problems



Even if…

- 😣 NOT reconfigurable! and/or
- 😣 many problems are **PSPACE**-complete!

Still want an "approximate" reconf. sequence
(e.g.) made up of almost-satisfying assignments

↓

RELAX feasibility to obtain approximate reconfigurability

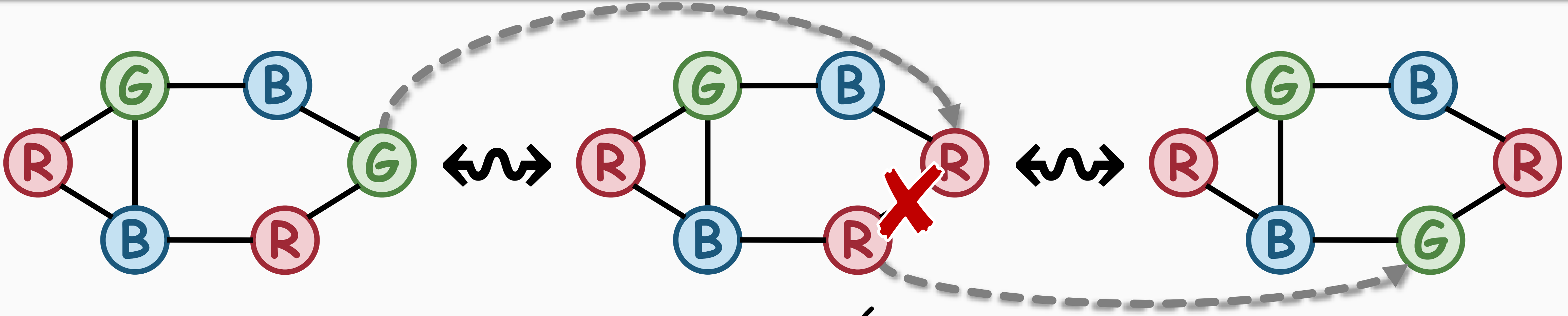
e.g. Set Cover Reconf. [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
Subset Sum Reconf. [Ito-Demaine. J. Comb. Optim. 2014]
Submodular Reconf. [O.-Matsuoka. WSDM 2022]

Example 2+

Maxmin q-CSP Reconfiguration

[IDHPSUU. Theor. Comput. Sci. 2011] [O. STACS 2023]

- **Input:** q-ary CSP $G = (V, E, \Sigma, (\pi_e)_{e \in E})$ & satisfying $\psi_{ini}, \psi_{tar}: V \to \Sigma$
- **Output:** $\psi = (\psi^{(1)} := \psi_{ini}, \dots, \psi^{(T)} := \psi_{tar})$ (reconf. sequence) s.t.
  ~~every $\psi^{(t)}$ satisfies all edges of G~~ (feasibility)
  $\mathrm{Ham}(\psi^{(t)}, \psi^{(t+1)}) \leq 1$ (adjacency)
- **Goal:** $\max_\psi \mathrm{val}_G(\psi) := \min_t$ (frac. of edges satisfied by $\psi^{(t)}$)



ψ_{ini} $\mathrm{val}_G(\psi) = {}^6/_7$ ψ_{tar}

$\text{Gap}_{1,1-\varepsilon}$ q-CSP Reconfiguration

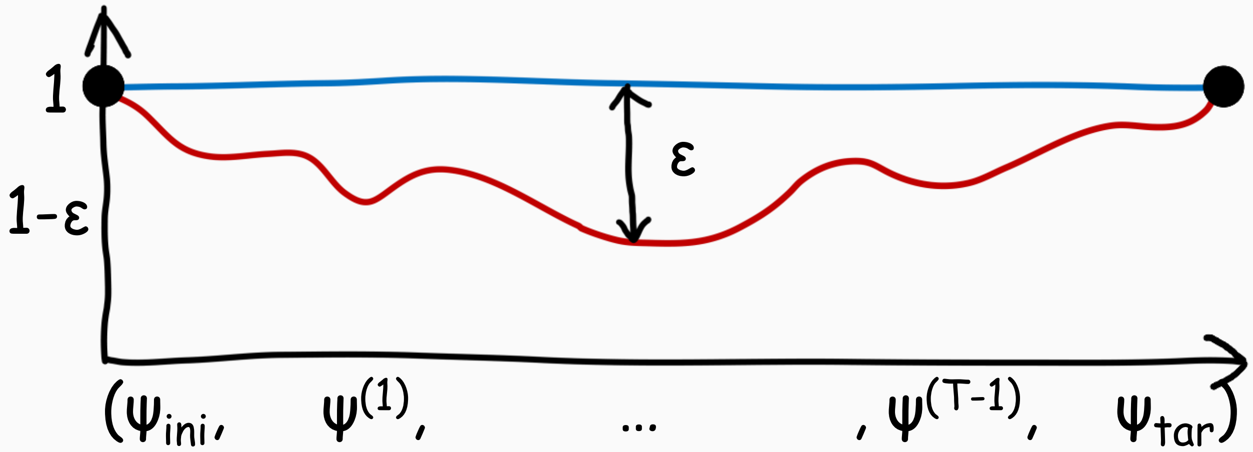
[IDHPSUU. Theor. Comput. Sci. 2011] [O. STACS 2023]

- **Input:** q-ary CSP $G = (V,E,\Sigma,(\pi_e)_{e\in E})$ & satisfying $\psi_{ini}, \psi_{tar}: V\to\Sigma$
- **Goal:** distinguish btw.

(Completeness) $\exists\psi\ \mathrm{val}_G(\psi) = 1$ (every $\psi^{(t)}$ satisfies all edges)

(Soundness) $\forall\psi\ \mathrm{val}_G(\psi) < 1-\varepsilon$ (some $\psi^{(t)}$ violates >ε-frac. of edges)

$\mathrm{val}_G(\psi) \coloneqq \min_t$ (frac. of edges satisfied by $\psi^{(t)}$)



"Reconf. analogue" of PCP theorem

- **Input:** q-ary CSP $G = (V,E,\Sigma,(\pi_e)_{e\in E})$ & satisfying $\psi_{ini}, \psi_{tar}: V\to\Sigma$
- **Goal:** distinguish btw.

(Completeness) $\exists\psi\ val_G(\psi) = 1$ (every $\psi^{(t)}$ satisfies all edges)

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- Reconfiguration **I**napproximability Hypothesis [O. STACS 2023]

"$\exists\varepsilon$, q, W: $Gap_{1,1-\varepsilon}$ q-CSP Reconf. with alphabet W is **PSPACE**-hard"

→😋Many reconf. problems are **PSAPCE**-hard to approx. conditionally

"Reconf. analogue" of PCP theorem

- **Input:** q-ary CSP $G = (V, E, \Sigma, (\pi_e)_{e \in E})$ & satisfying $\psi_{ini}, \psi_{tar}: V \to \Sigma$
- **Goal:** distinguish btw.

(Completeness) $\exists \psi\ val_G(\psi) = 1$ (every $\psi^{(t)}$ satisfies all edges)

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- Probabilistically Checkable Reconfiguration Proof (PCRP) theorem
  [Hirahara-O. STOC 2024] [Karthik C. S.-Manurangsi. 2023]

  ∃ε, q, W: $Gap_{1,1\text{-}\varepsilon}$ q-CSP Reconf. with alphabet W is **PSPACE**-hard

  ➔😋Many reconf. problems are **PSAPCE**-hard to approx. unconditionally

⚠ Still do not know explicit values of ε, q, W

Toward a better trade-off btw. ε, q, W…?

😖 := "any large or small" const.

ε := gap;   q := query complexity;   W := alphabet size

ε	q	W
😖	😖	😖

PCRP theorem
[Hirahara-O. STOC 2024]
[Karthik C. S.-Manurangsi. 2023]

Gap reduction
[O. STACS 2023]

ε	q	W
😖	2	3

Serial repetition

ε	q	W
$\frac{1}{2}$	😖	3

ε	q	W
0.001	2	😖

Gap amplification
[O. SODA 2024]

ε	q	W
10^{-18}	2	$2 \cdot 10^6$

😋😋😋😋😋
Alphabet reduction
(this paper)

Our contribution

Alphabet reduction à la [Dinur. J. ACM 2007]

PCRP · Gap 2-CSP Reconf.	
soundness error	: $1 - \varepsilon$
query complexity	: 2
alphabet size	: W

→

PCRP · Gap 2-CSP Reconf.	
soundness error	: $1 - \kappa \cdot \varepsilon$
query complexity	: 2
alphabet size	: $W_0 := 2 \cdot 10^6$

- Reduce ANY BIG W to UNIVERSAL W_0 preserving ε by κ-factor
- ε can be $o(1)$
  unlike degree reduction [O. STACS 2023] & gap amplification [O. SODA 2024]

Our contribution
Consequences

- "Weak" PCRP for **PSPACE** with any small ε & large q, W

[O. STACS 2023]
[O. SODA 2024]
(this paper)

😋 Independent of ε, q, W

- PCRP for **PSPACE** with $\varepsilon_0 = 10^{-18}$, $q_0 = 2$, $W_0 = 2 \cdot 10^6$

[O. STACS 2023]
[O. SODA 2024]
(this paper)

2-CSP Reconf, 3-SAT Reconf, Independent Set Reconf, Vertex Cover Reconf, Clique Reconf, Dominating Set Reconf, Nondeterministic Constraint Logic

are **PSPACE**-hard to approximate within a factor of $1-\delta_0$

Proof sketch

Robustization - Main challenge

Maxmin 2-CSP Reconf.

$G = (V, E, \Sigma, (\pi_e)_{e \in E})$

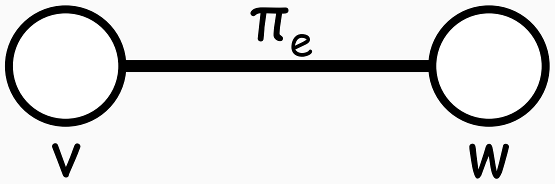
ψ_{ini} & ψ_{tar}: $V \to \Sigma$

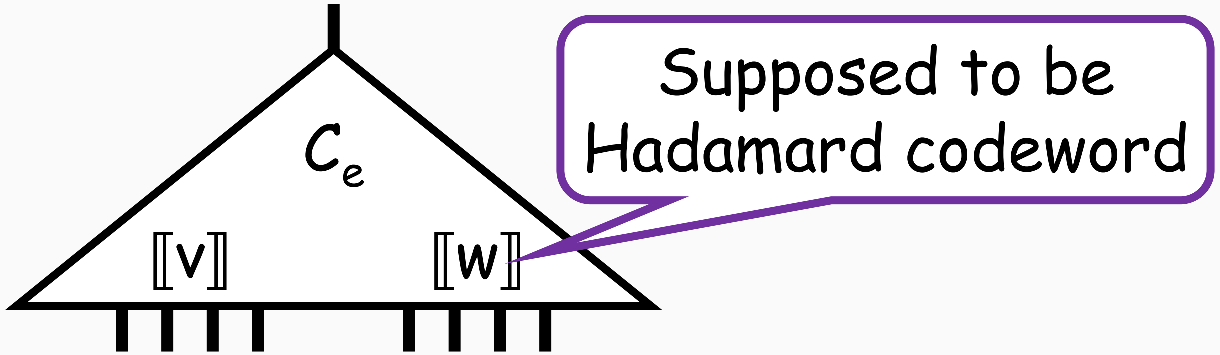
→

Circuit SAT Reconf.

$\mathcal{C} = (C_e)_{e \in E}$ where C_e: $\mathbb{F}_2^{\ell \times 2} \to \mathbb{F}_2$

σ_{ini} & σ_{tar}: $V \to \mathbb{F}_2^{\ell}$





🎯 (**Perfect** completeness)

$\exists \psi$ $val_G(\psi) = 1$ $\Longrightarrow$ $\exists \boldsymbol{\sigma}$ $\forall \sigma^{(t)}$ satisfies $\mathcal{C}$

🎯 (**Robust** soundness)

$\forall \psi$ $val_G(\psi) < 1\text{-}\varepsilon$ $\Longrightarrow$ $\forall \boldsymbol{\sigma}$ $\exists \sigma^{(t)}$ s.t. asgmt. for ε-frac of C_e is **.01%-far** from satisfying asgmt.

Proof sketch

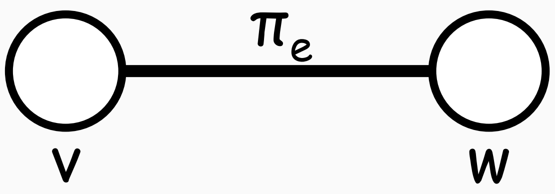
Robustization - Main challenge

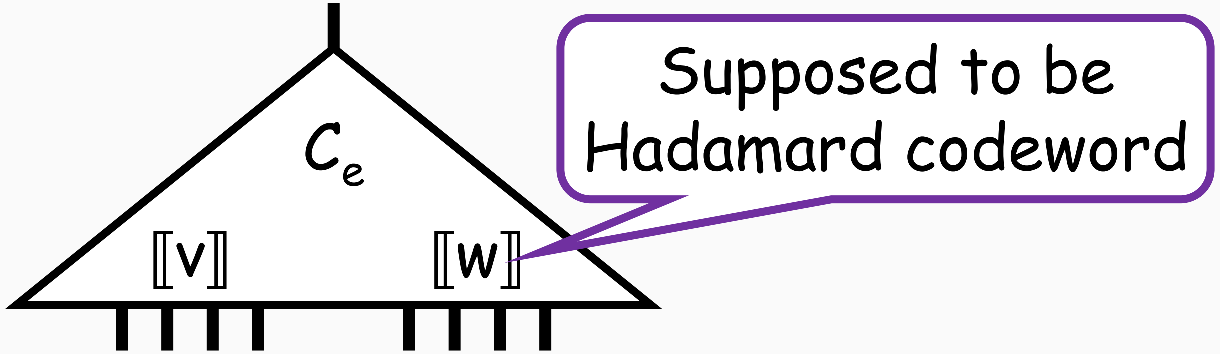
Maxmin 2-CSP R... ...T Reconf.

$G = (V, E, \ldots$ ... $\mathbb{F}_2^{\ell \times 2} \to \mathbb{F}_2$

ψ_{ini} & ψ_{tar}: V ... $\mathbb{F}_2^{\ell}$

Q. How to design C_e's?





🎯 (**Perfect** completeness)

$\exists\psi\ val_G(\psi) = 1 \implies \exists\sigma\ \forall\sigma^{(t)}$ satisfies $\mathcal{C}$

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$\forall\psi\ val_G(\psi) < 1\text{-}\varepsilon \implies \forall\sigma\ \exists\sigma^{(t)}$ s.t. asgmt. for ε-frac of C_e is **.01%-far** from satisfying asgmt.

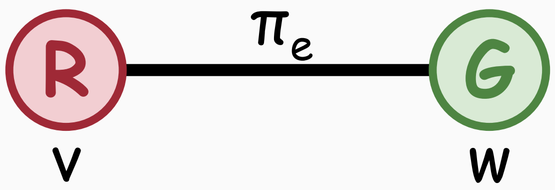
Proof sketch

Failed attempt 1: Perfect completeness fails 😭

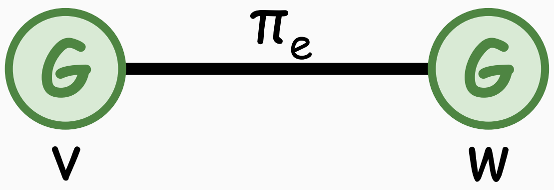
G is edge $e = (v,w)$

- $\Sigma := \{R, G\}$
- $\pi_e := \Sigma \times \Sigma$ (always satisfied)

$\psi_{ini} := (R, G)$



Trivially...



$\psi_{tar} := (G, G)$

$C_e(f \circ g) = 1 \Leftrightarrow \exists \alpha, \beta \in \Sigma$ s.t.

- $f \circ g = \mathrm{Had}(\alpha) \circ \mathrm{Had}(\beta)$
- $(\alpha, \beta) \in \pi_e$

$\sigma_{ini} := \mathrm{Had}(R) \circ \mathrm{Had}(G)$

?

$\sigma_{tar} := \mathrm{Had}(G) \circ \mathrm{Had}(G)$

Proof sketch

Failed attempt 1: Perfect completeness fails 😭

G is edge $e = (v,w)$

- $\Sigma := \{R, G\}$
- $\pi_e := \Sigma \times \Sigma$ (always satisfied)

$C_e(f \circ g) = 1 \iff \exists \alpha, \beta \in \Sigma$ s.t.

- $f \circ g = \mathrm{Had}(\alpha) \circ \mathrm{Had}(\beta)$
- $(\alpha, \beta) \in \pi_e$

ACTUALLY…

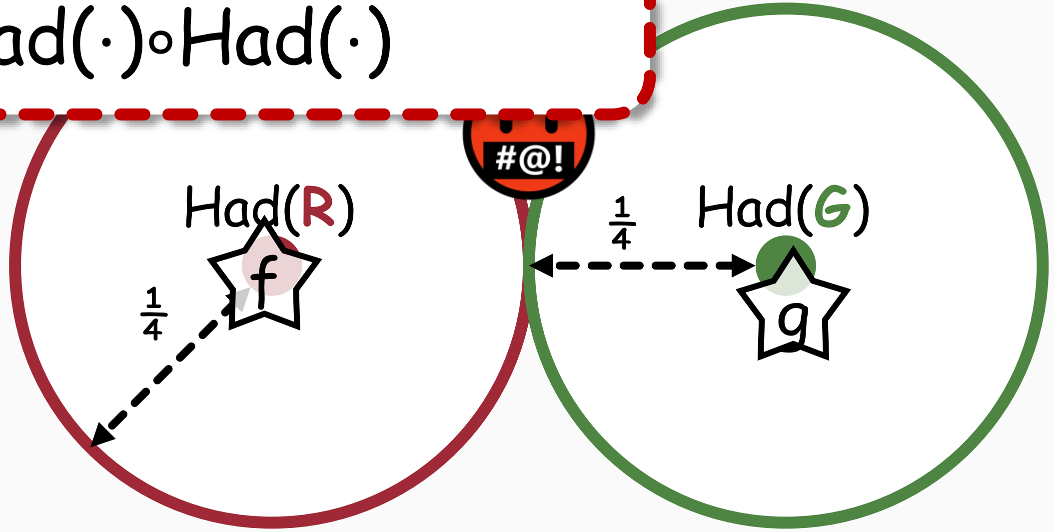
$\forall \sigma = (f^{(1)} \circ g^{(1)}, \ldots, f^{(T)} \circ g^{(T)})$ from σ_{ini} to σ_{tar}

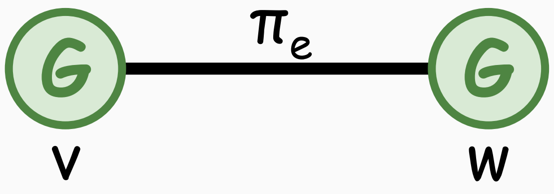
$\exists f^{(t)} \circ g^{(t)}$ is $\frac{1}{8}$-far from $\mathrm{Had}(\cdot) \circ \mathrm{Had}(\cdot)$

$\psi_{ini} := (R, G)$

Had(G)

Trivially…





$\psi_{tar} := (G, G)$

$\sigma_{tar} := \mathrm{Had}(G) \circ \mathrm{Had}(G)$

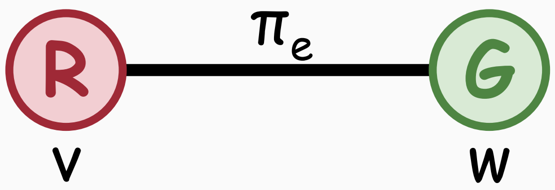
Proof sketch

Failed attempt 2: Robust soundness fails 😭

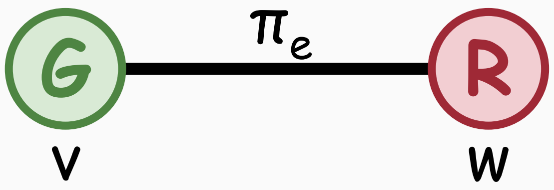
G is edge $e = (v,w)$

- $\Sigma := \{R, G\}$
- $\pi_e := \{(R, G), (G, R)\}$ (bichromatic)

$\psi_{ini} := (R, G)$



$\psi_{tar} := (G, R)$



$C_e(f \circ g) = 1 \Leftrightarrow$

- f & g are $\frac{1}{4}$-close to Had(·)
- $\Delta(f, \mathrm{Had}(\boldsymbol{\alpha})) \leq \frac{1}{4}$ & $\Delta(g, \mathrm{Had}(\boldsymbol{\beta})) \leq \frac{1}{4}$

$\Longrightarrow (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \pi_e$

$\sigma_{ini} := \mathrm{Had}(R) \circ \mathrm{Had}(G)$

?

$\sigma_{tar} := \mathrm{Had}(G) \circ \mathrm{Had}(R)$

Proof sketch

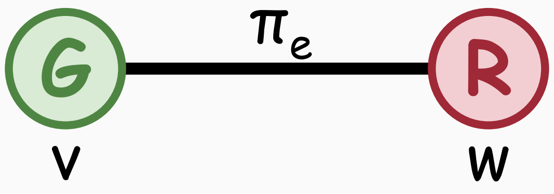
Failed attempt 2: Robust soundness fails 😭

G is edge $e = (v,w)$

- $\Sigma := \{R, G\}$
- $\pi_e := \{(R, G), (G, R)\}$ (bichromatic)

$\psi_{ini} := (R,$ ...

$\psi_{tar} := (G, R)$



$C_e(f \circ g) = 1 \Leftrightarrow$

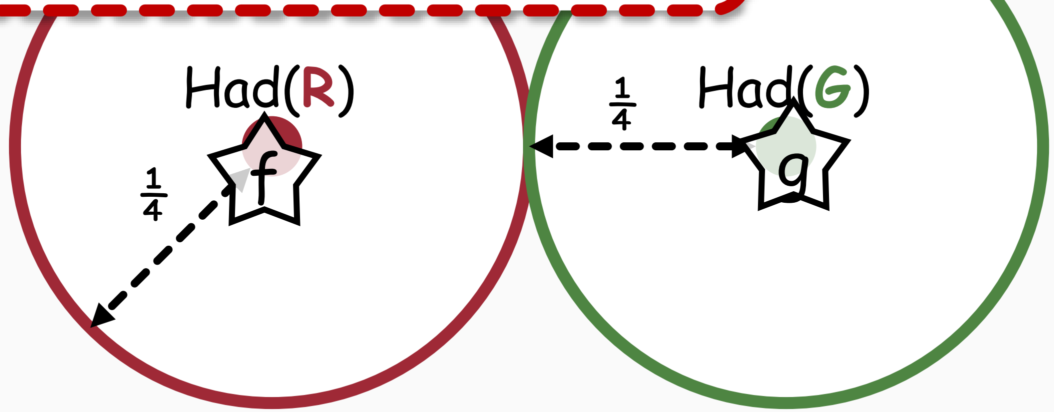
- f & g are $\frac{1}{4}$-close to Had$(\cdot)$
- $\Delta(f, \mathrm{Had}(\alpha)) \le \frac{1}{4}$ & $\Delta(g, \mathrm{Had}(\beta)) \le \frac{1}{4}$

...ad(G))

ACTUALLY...

$\exists \boldsymbol{\sigma} = (f^{(1)} \circ g^{(1)}, \ldots, f^{(T)} \circ g^{(T)})$ from σ_{ini} to σ_{tar}

$\forall f^{(t)} \circ g^{(t)}$ is $o(1)$-close to satisfying asgmt. of C_e



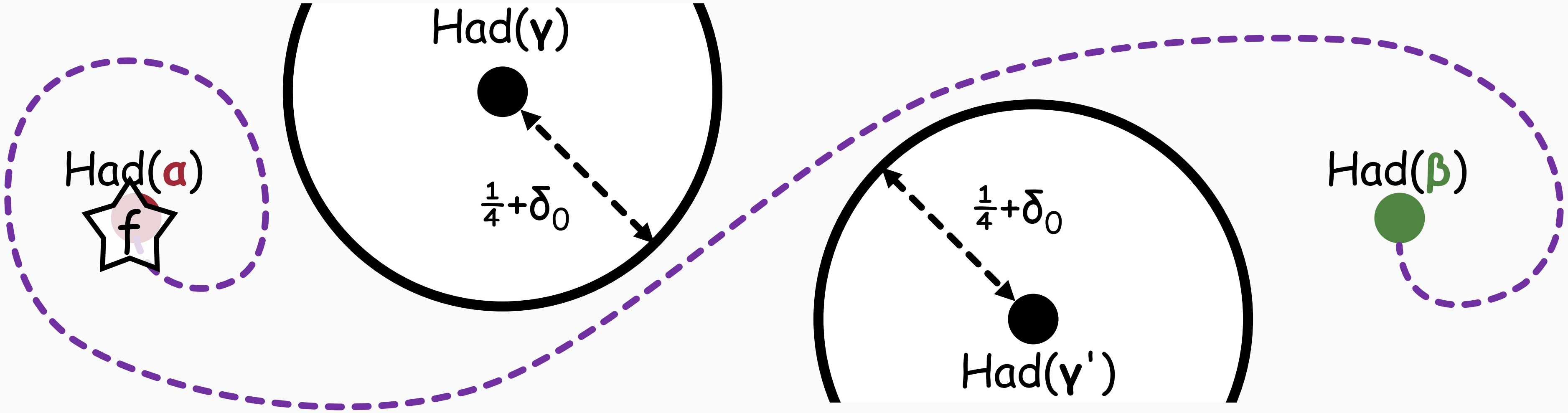
$\sigma_{tar} := \mathrm{Had}(G) \circ \mathrm{Had}(R)$

Our solution

Reconfigurability of Hadamard codes

$\forall \alpha \neq \beta \in \mathbb{F}_2^n \quad \exists \mathbf{f} = (f^{(1)}, \ldots, f^{(T)})$ from $\mathrm{Had}(\alpha)$ to $\mathrm{Had}(\beta)$ s.t.

- $\min \{ \Delta(f^{(t)}, \mathrm{Had}(\alpha)), \ \Delta(f^{(t)}, \mathrm{Had}(\beta)) \} \leq \frac{1}{4}$
- $\forall \gamma \neq \alpha, \beta \quad \Delta(f^{(t)}, \mathrm{Had}(\gamma)) > \frac{1}{4} + \delta_0 \quad (\delta_0 = 0.01)$



😋Can reconfigure btw. Hadamard codewords without getting too close to the other codewords

Conclusions

- 🔁 Alphabet reduction for 2-CSP Reconf. à la [Dinur. J. ACM 2007]
- Make gap ε & alphabet size W oblivious to parameters of PCRPs
  [Hirahara-Ohsaka. STOC 2024] [Karthik C. S.-Manurangsi. 2023]
- Optimal trade-off btw. ε, q, W?
- Other applications of
  Reconfigurability of Hadamard codes?