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Optimal PSPACE-hardness of Approximating Set Cover Reconfiguration

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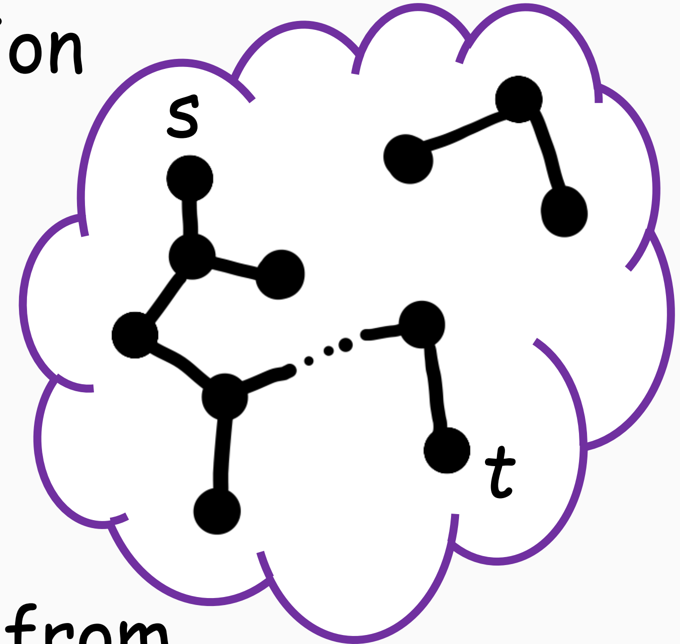
Intro of reconfiguration

Imagine **connecting** a pair of feasible solutions (of NP problem)
under a particular adjacency relation



Q. Is a pair of solutions reachable to each other?
Q. If so, what is the shortest transformation?
Q. If not, how can the feasibility be relaxed?

Many reconfiguration problems have been derived from

Satisfiability, Coloring, Vertex Cover, Clique, Dominating Set, Feedback Vertex Set, Steiner Tree, Matching, Spanning Tree, Shortest Path, Set Cover, Subset Sum, …

See [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011] [Nishimura. Algorithms 2018] [van den Heuvel. Surv. Comb. 2013] [Hoang. https://reconf.wikidot.com/]

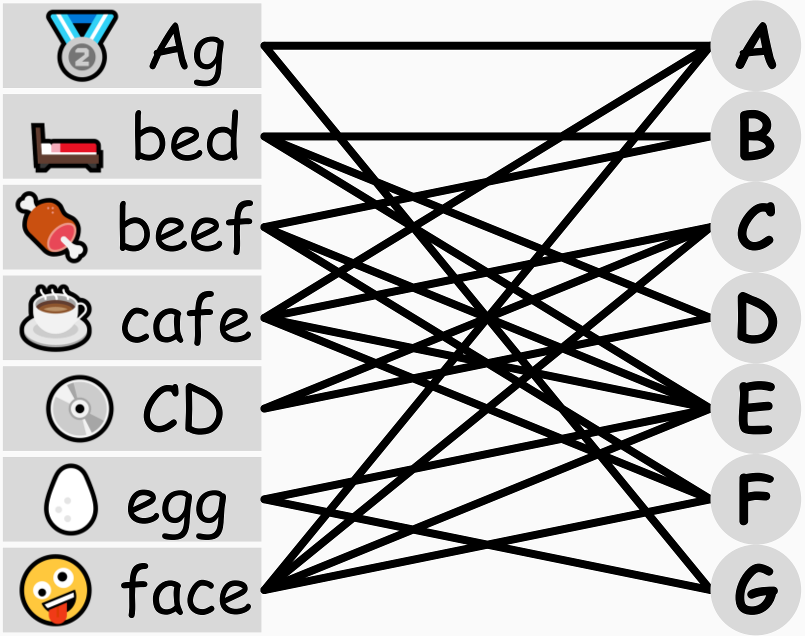
Example 1

Set Cover Reconfiguration

- **Input:** Set system $\mathscr{F}$ & covers C_{start} & C_{goal} of size k
- **Output:** $\mathcal{C} = (C^{(1)} := C_{start}, \ldots, C^{(T)} := C_{goal})$ (reconf. sequence) s.t.
  $C^{(t)}$ covers $\mathscr{F}$ & $|C^{(t)}| \leq k+1$ (feasibility)
  $|C^{(t)} \triangle C^{(t+1)}| \leq 1$ (adjacency)
- YES case $(k = 3)$

Ag
bed
cafe

C_{start}



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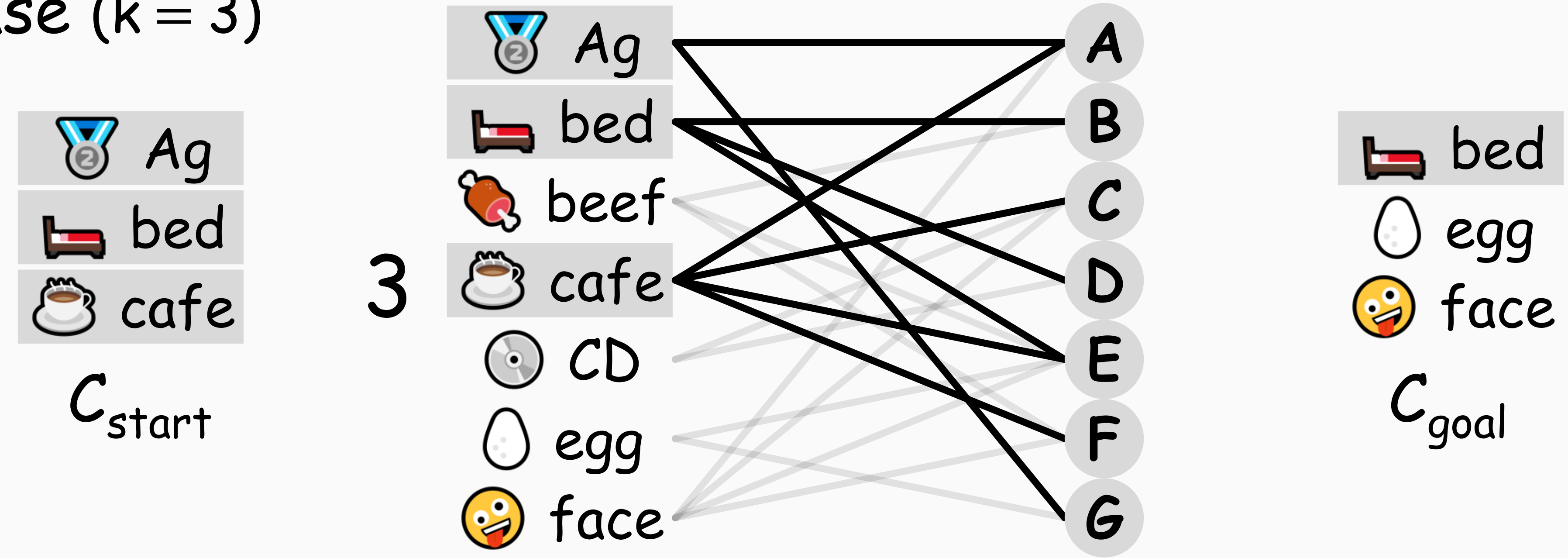

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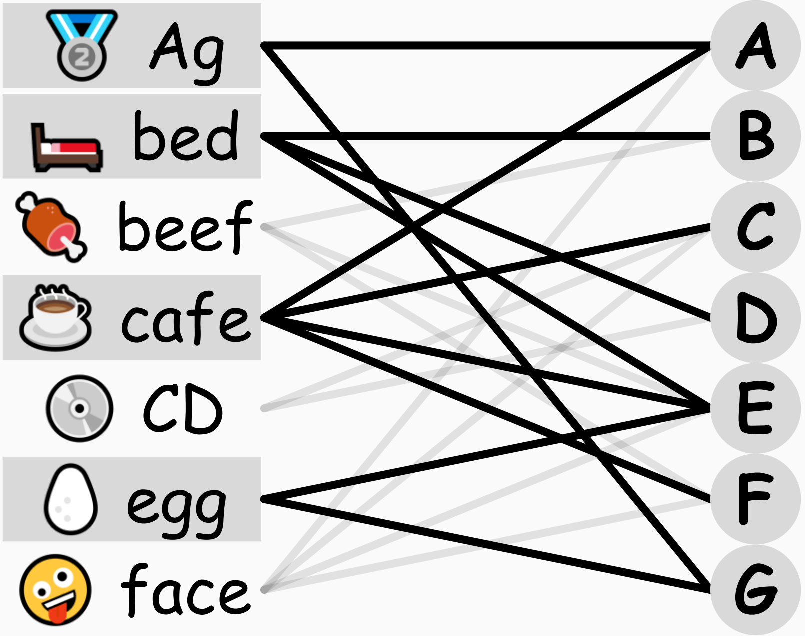
cafe

C_{start}

4



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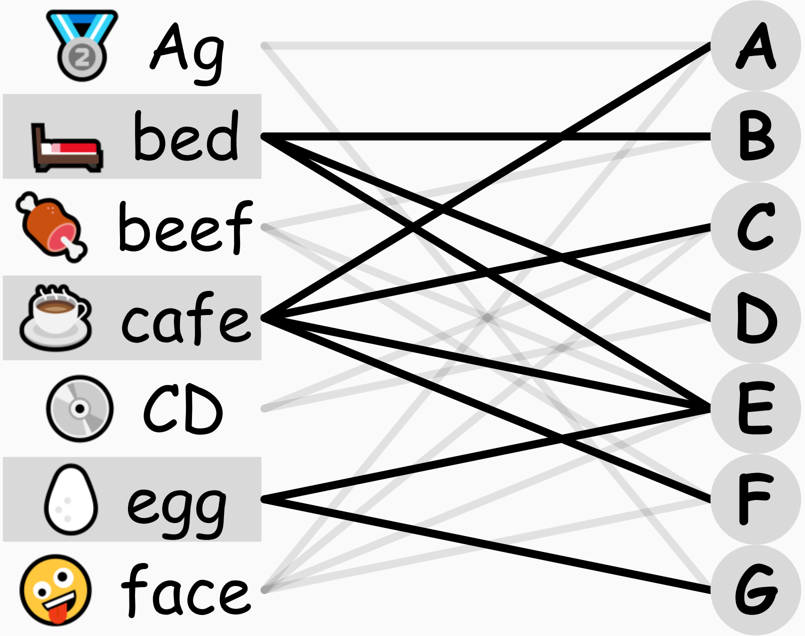
3



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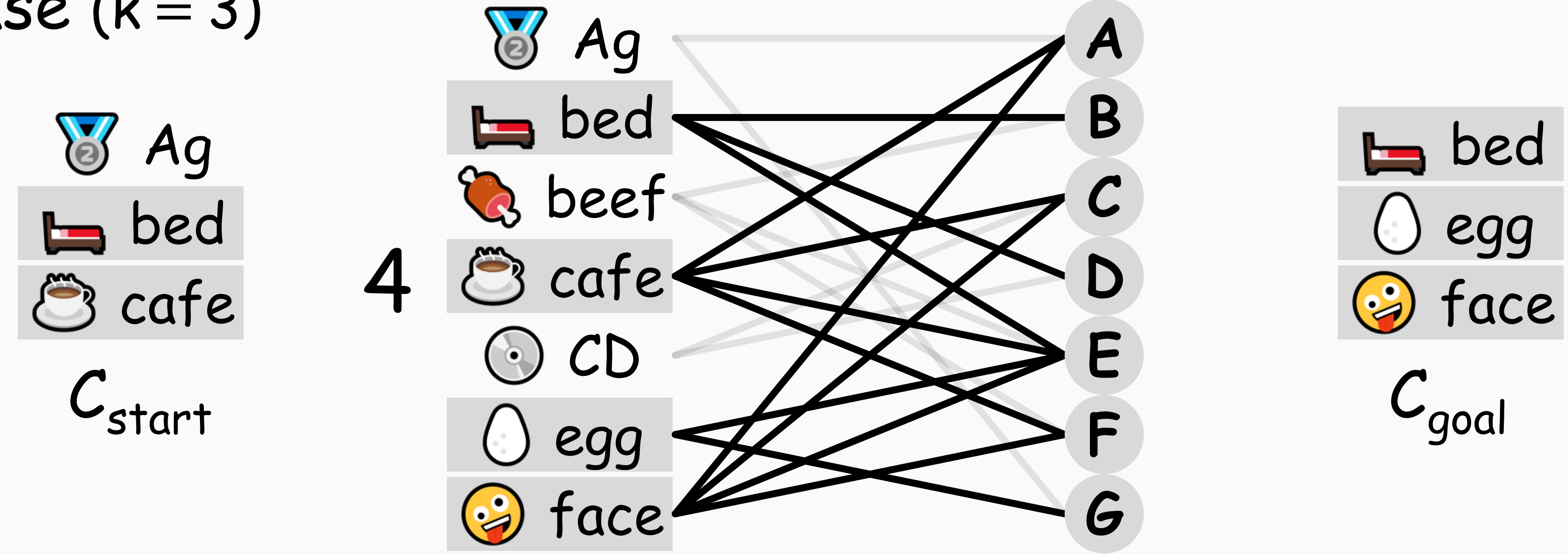

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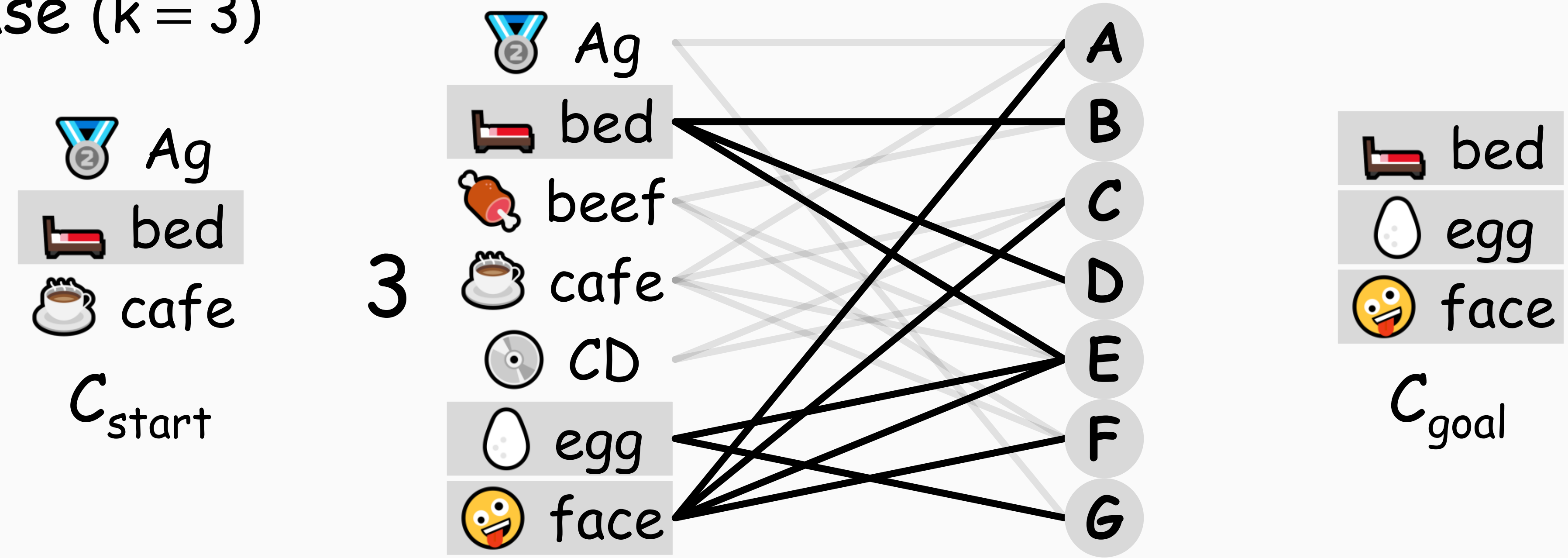

Example 2

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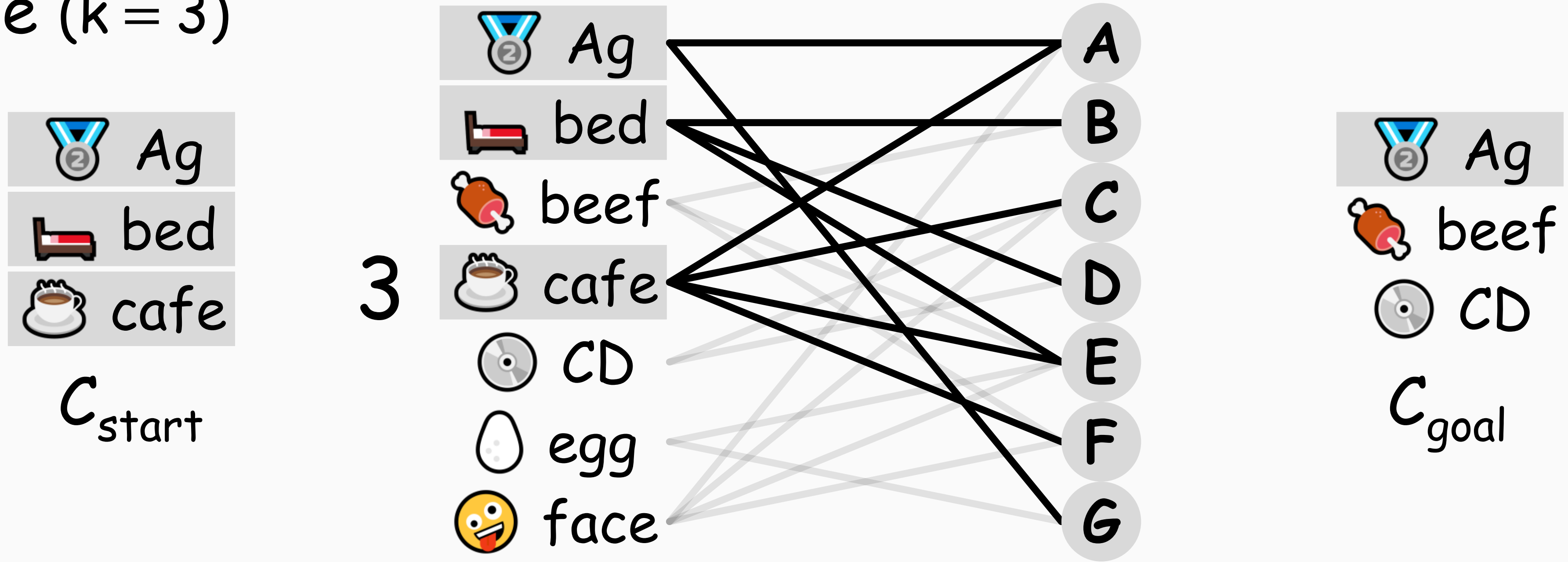

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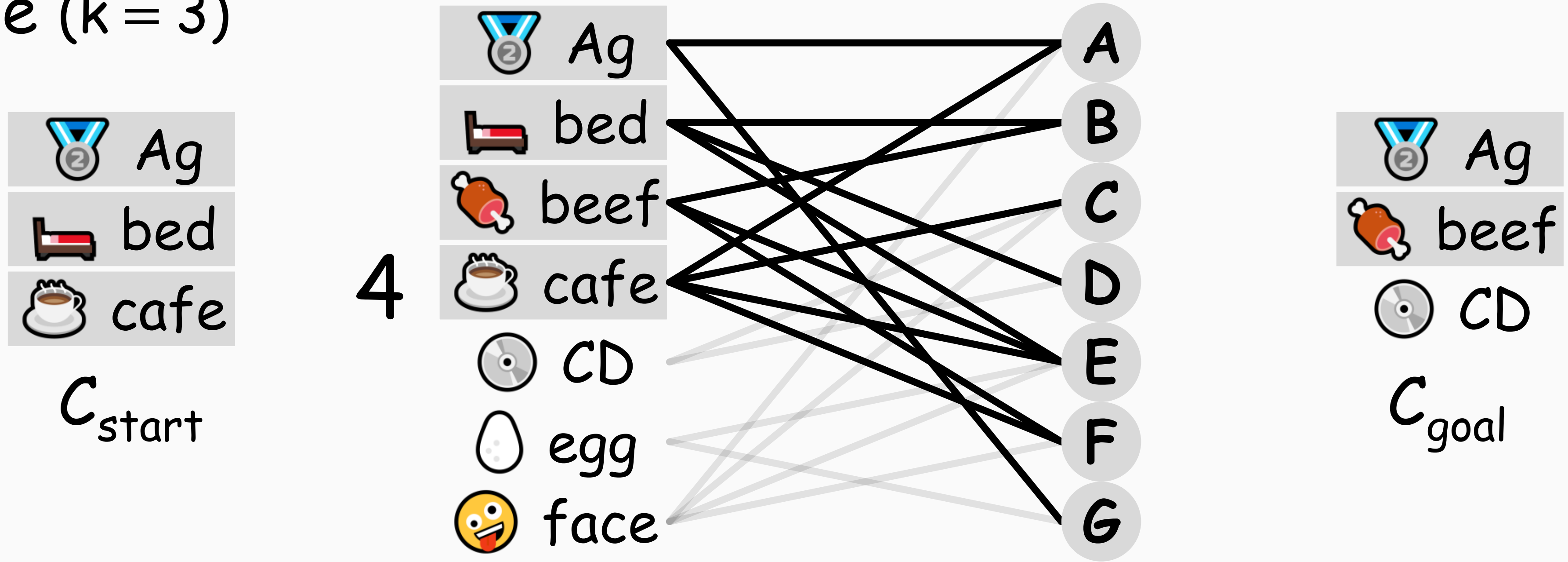

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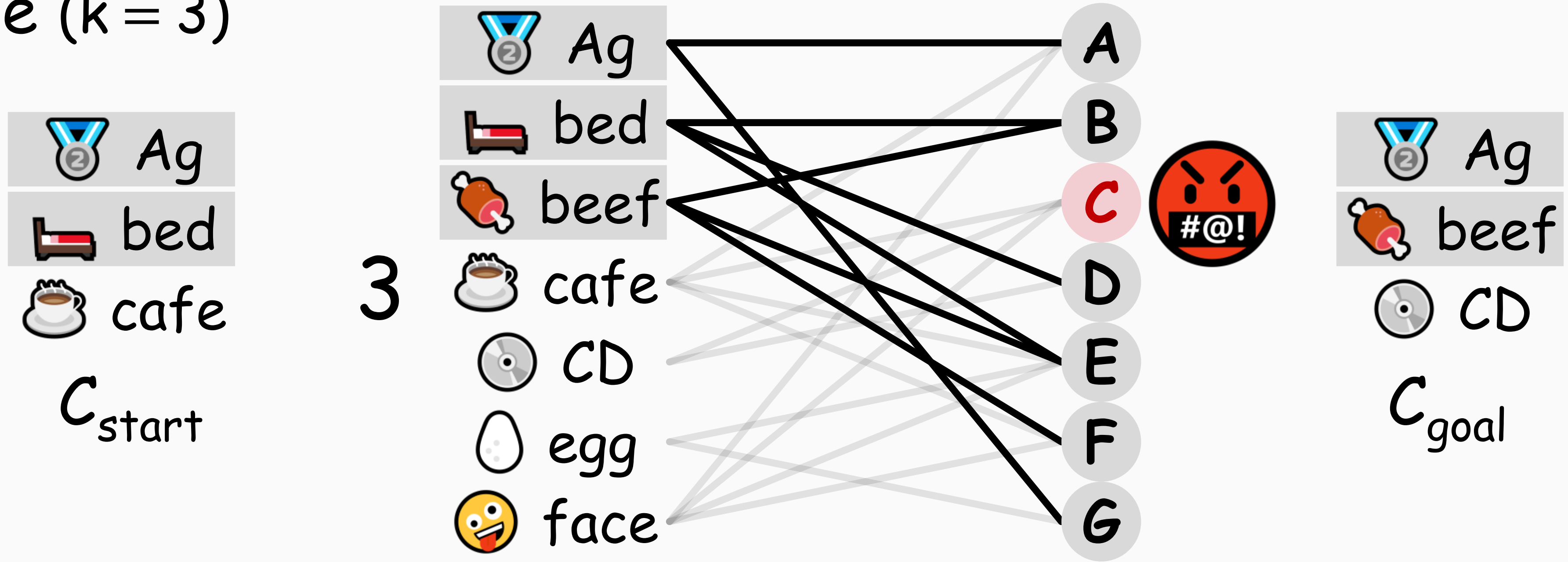
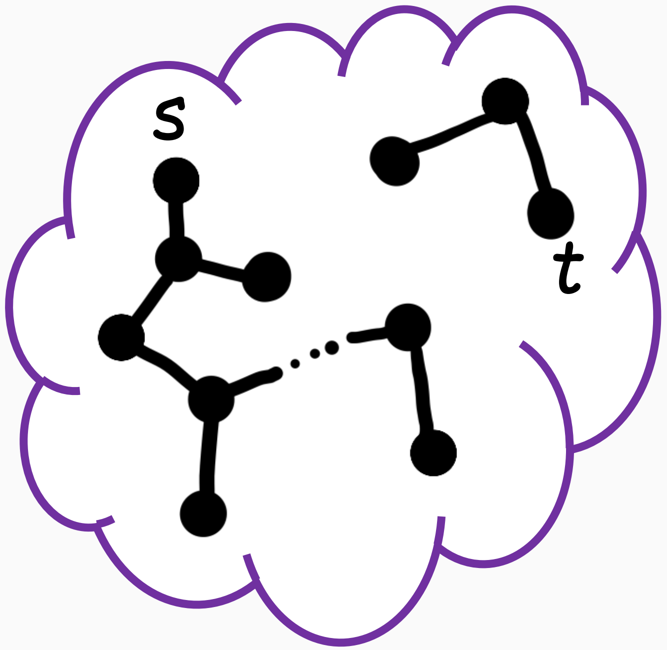


Optimization versions of reconfiguration problems

Even if…

- 😖NOT reconfigurable! and/or
- 😖many problems are **PSPACE-complete**!



Still want an "approximate" reconf. sequence

(e.g.) made up of not-too-large set covers

↓

RELAX feasibility to obtain approximate reconfigurability

e.g. Set Cover Reconf. [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Subset Sum Reconf. [Ito-Demaine. J. Comb. Optim. 2014]

Submodular Reconf. [O.-Matsuoka. WSDM 2022]

Example 2+

Minmax Set Cover Reconfiguration

- **Input:** Set system $\mathscr{F}$ & covers C_{start} & C_{goal} of size k
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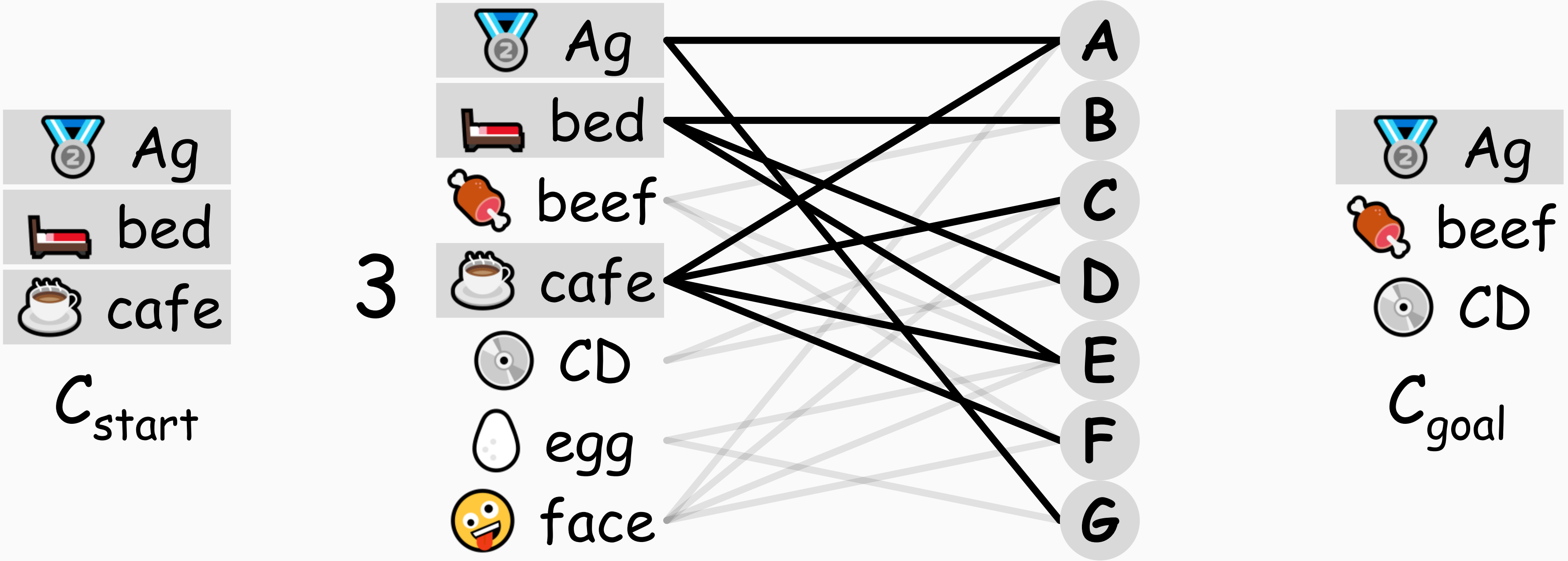
- **Goal:** $\min_{\mathcal{C}} \mathrm{val}_{\mathscr{F}}(\mathcal{C}) := \max_t |C^{(t)}|$



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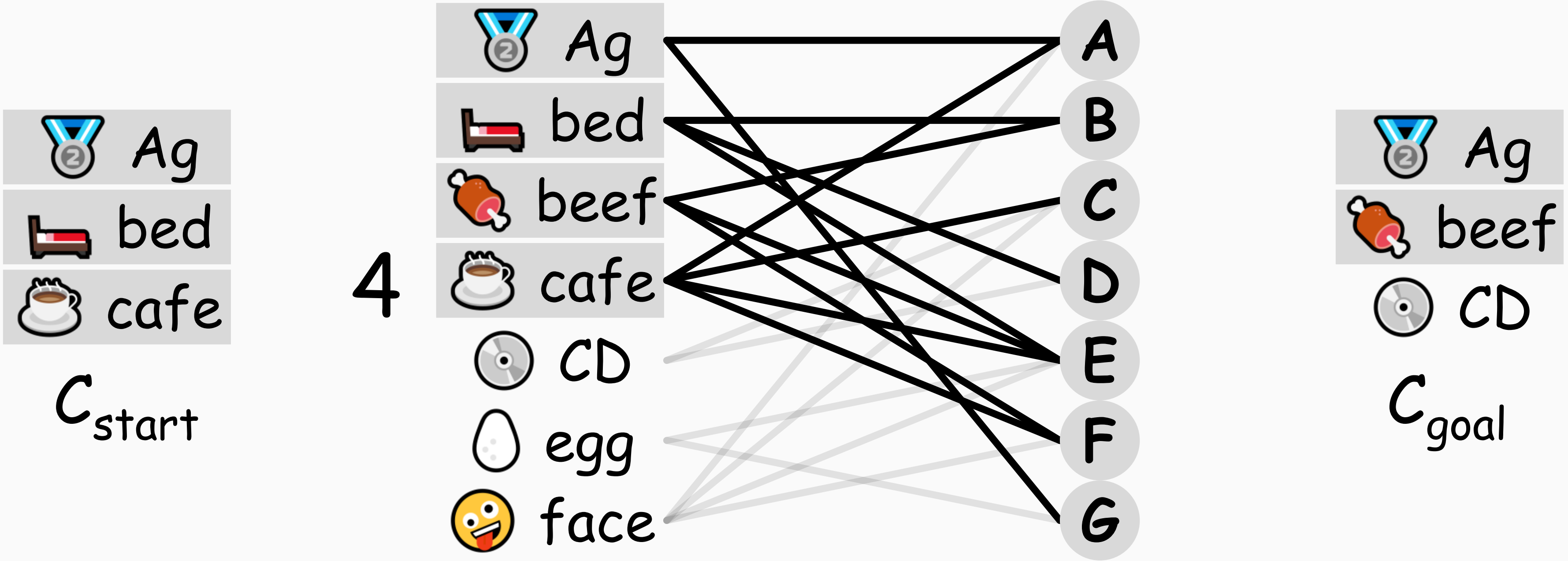

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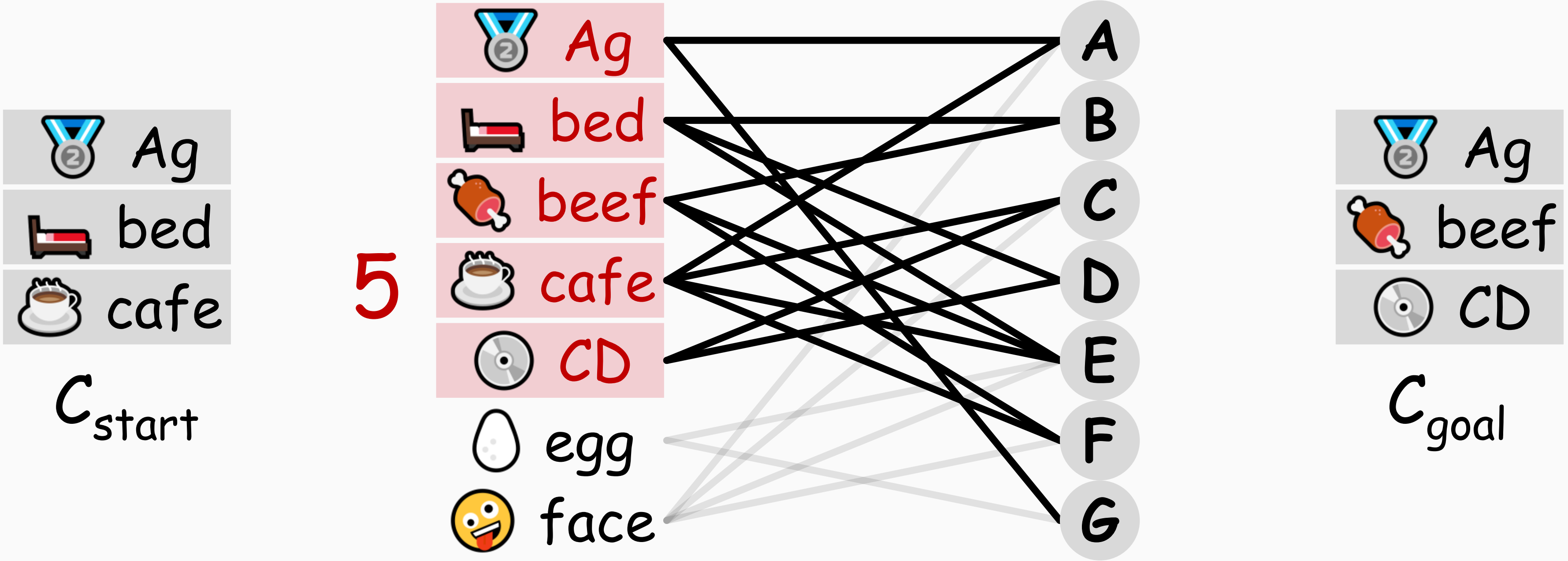

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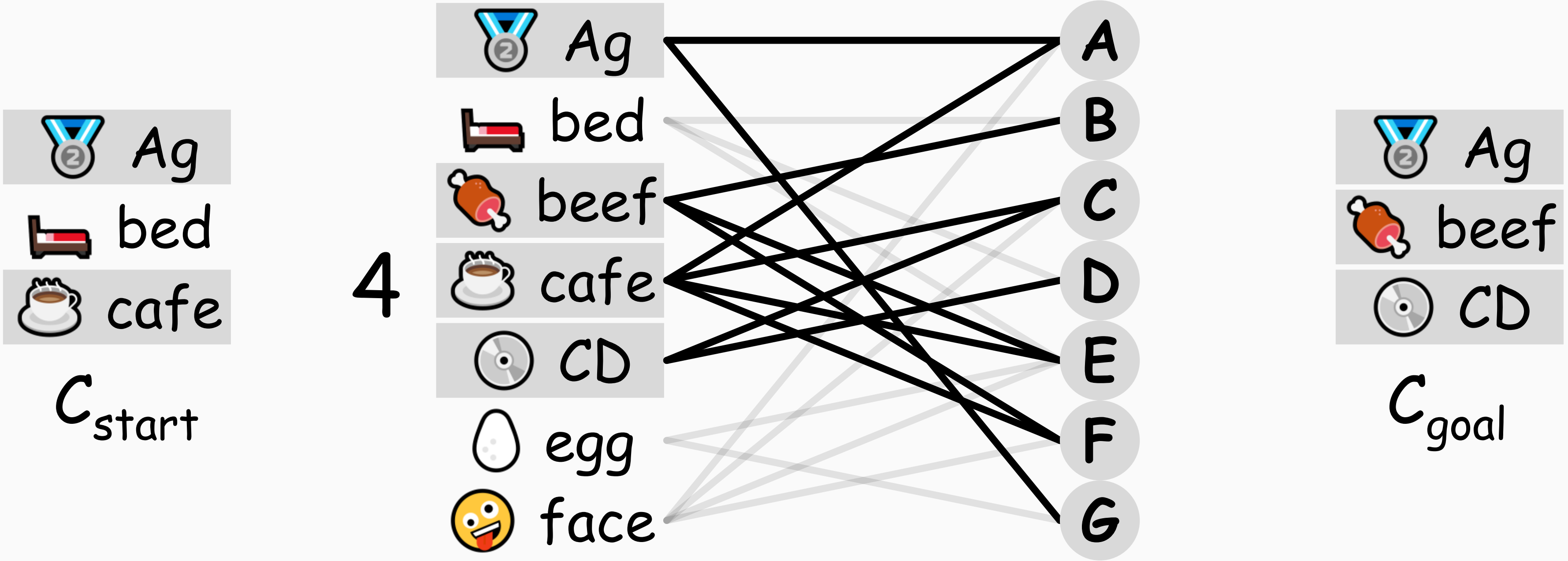

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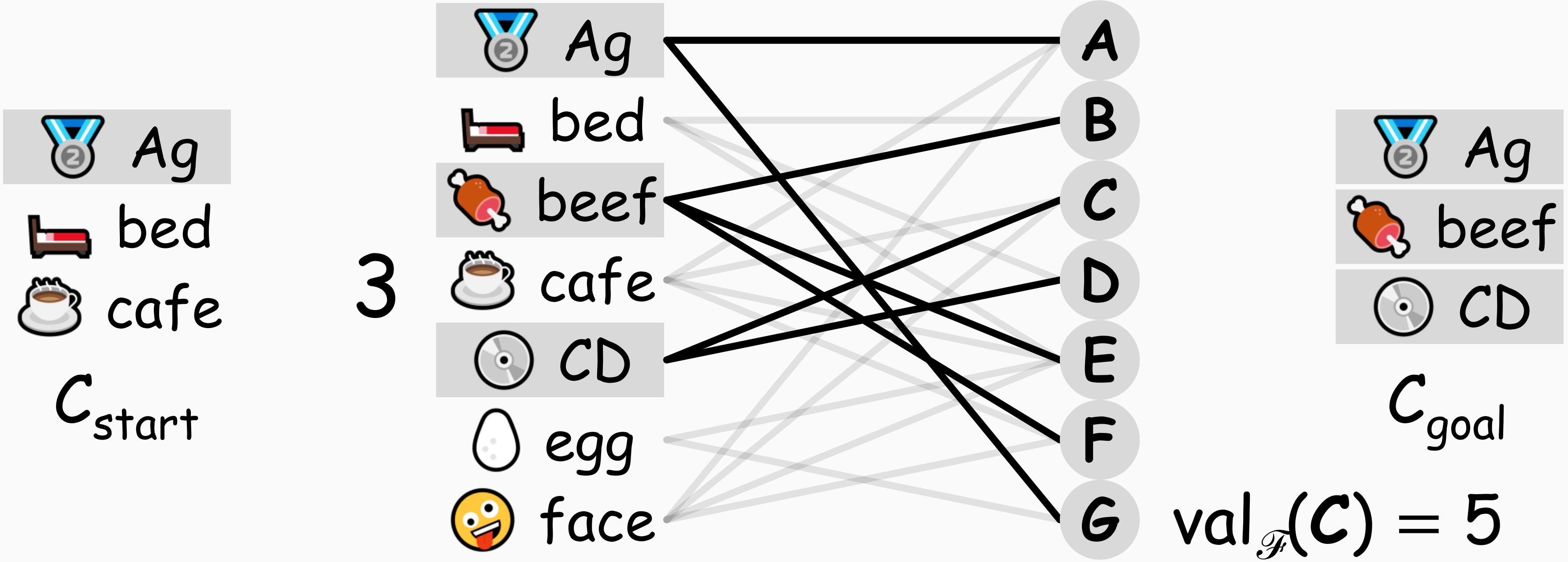
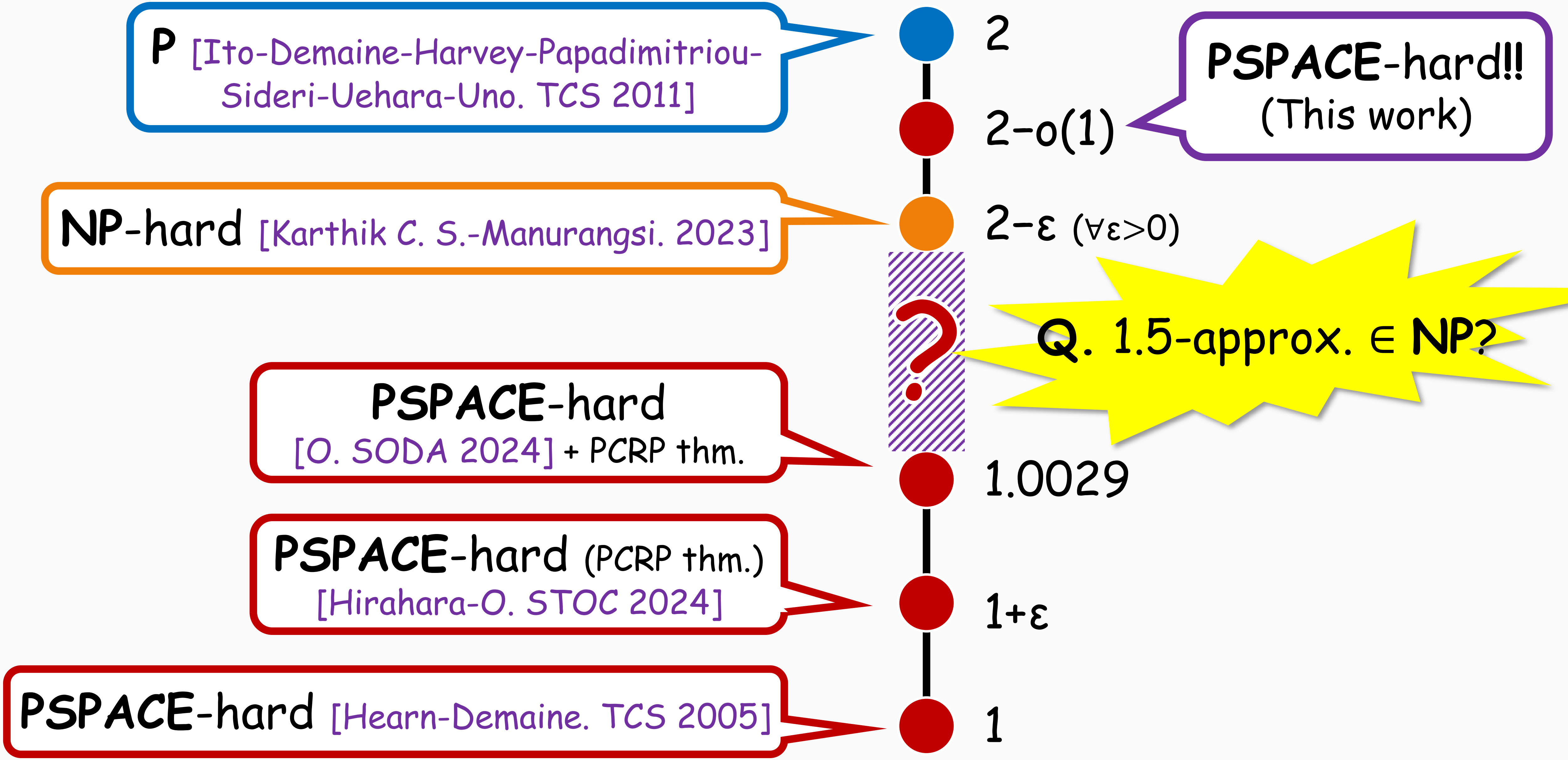


Known results on Minmax Set Cover Reconf.



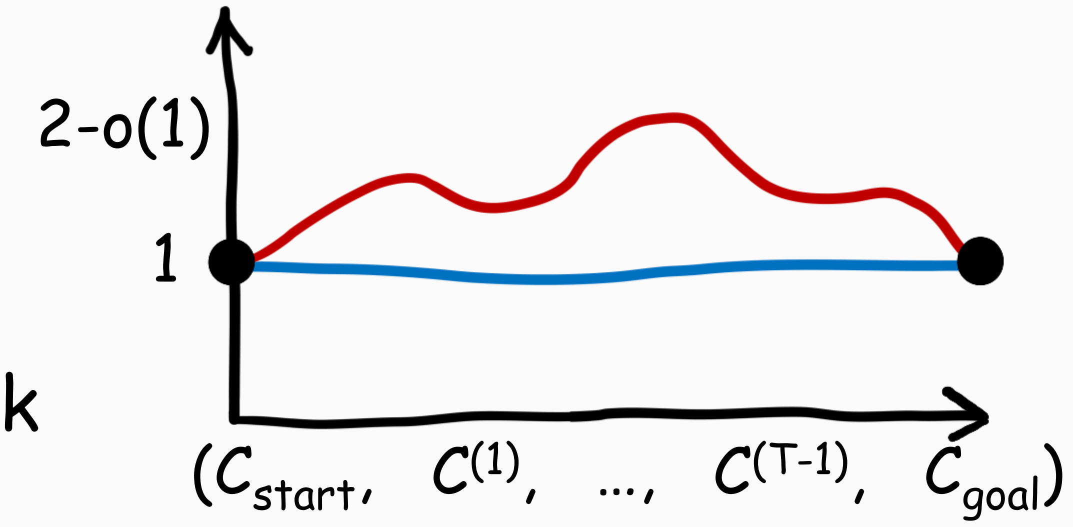
Known results on Minmax Set Cover Reconf.

- 2 — P [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. TCS 2011]
- 2−o(1) — PSPACE-hard!! (This work)
- 2−ε (∀ε>0) — NP-hard [Karthik C. S.-Manurangsi. 2023]

The main open question is clear: *Can we prove tight* PSPACE*-hardness of approximation results for GapMaxMin-2-CSP*$_q$ *and Set Cover Reconfiguration?*

- 1.0029 — PSPACE-hard [O. SODA 2024] + PCRP thm.
- 1+ε — PSPACE-hard (PCRP thm.) [Hirahara-O. STOC 2024]
- 1 — PSPACE-hard [Hearn-Demaine. TCS 2005]

Our contribution



- **Input:** Set system $\mathscr{F}$
  Covers C_{start} & C_{goal} of size k

PSPACE-hard to distinguish between

(Completeness) ∃reconf. sequence ∀cover has size $\leq$ k+1

(Soundness) ∀reconf. sequence ∃cover has size $>$ $(2-o(1))\cdot(k+1)$

➜😋 Minmax Set Cover Reconfiguration is **PSPACE**-hard to approx. within 2–o(1)

👆FIRST sharp approx. threshold

for reconf. problems

Related work

- Min Set Cover

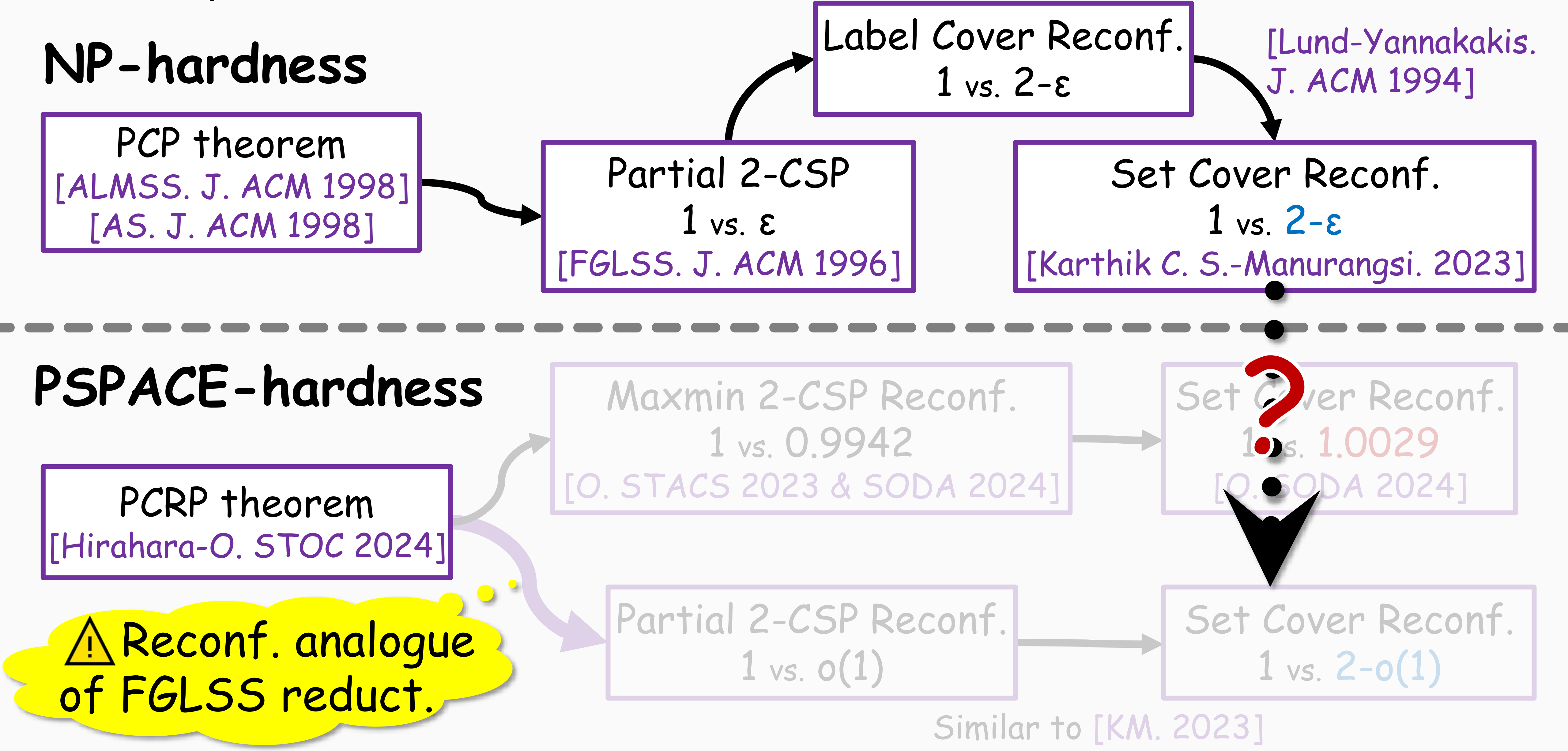
  ln N-approx. in **P** [Johnson. J. Comput. System Sci. 1974] [Lovász. Discrete Math. 1975]

  $(1-\varepsilon)\cdot\ln N$ is **NP**-hard [Feige. J. ACM 1998] [Dinur-Steurer. STOC 2014]

- **PSPACE**-hardness of approx. for reconfiguration problems

Clique Reconf.	n^{ε}-approx.	[Hirahara-O. STOC 2024]
2-CSP Reconf.	0.9942-approx.	[O. SODA 2024] [O. ICALP 2024]
many problems	$(1+\varepsilon)$-approx.	[O. STACS 2023] [Hirahara-O. STOC 2024]

Proof outline



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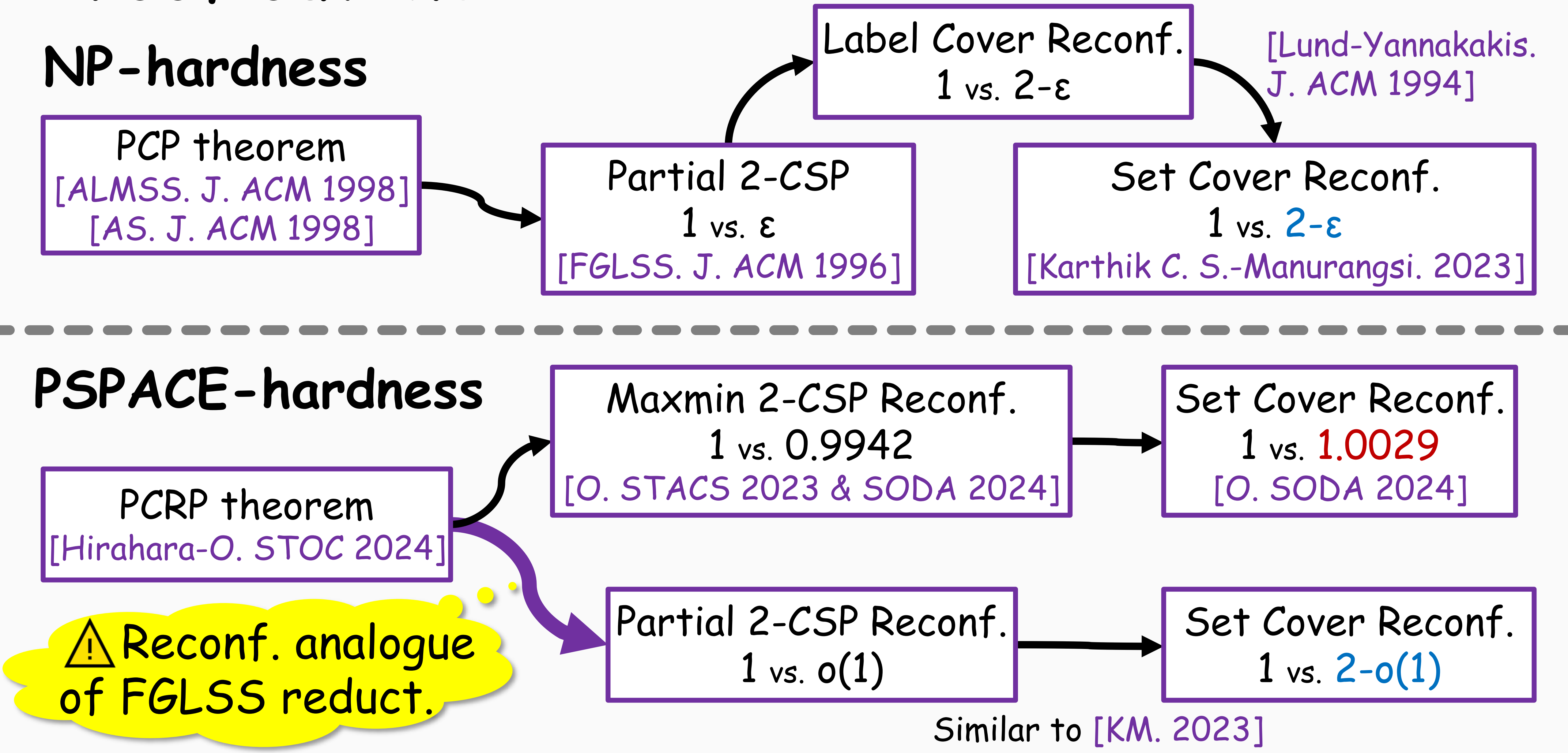


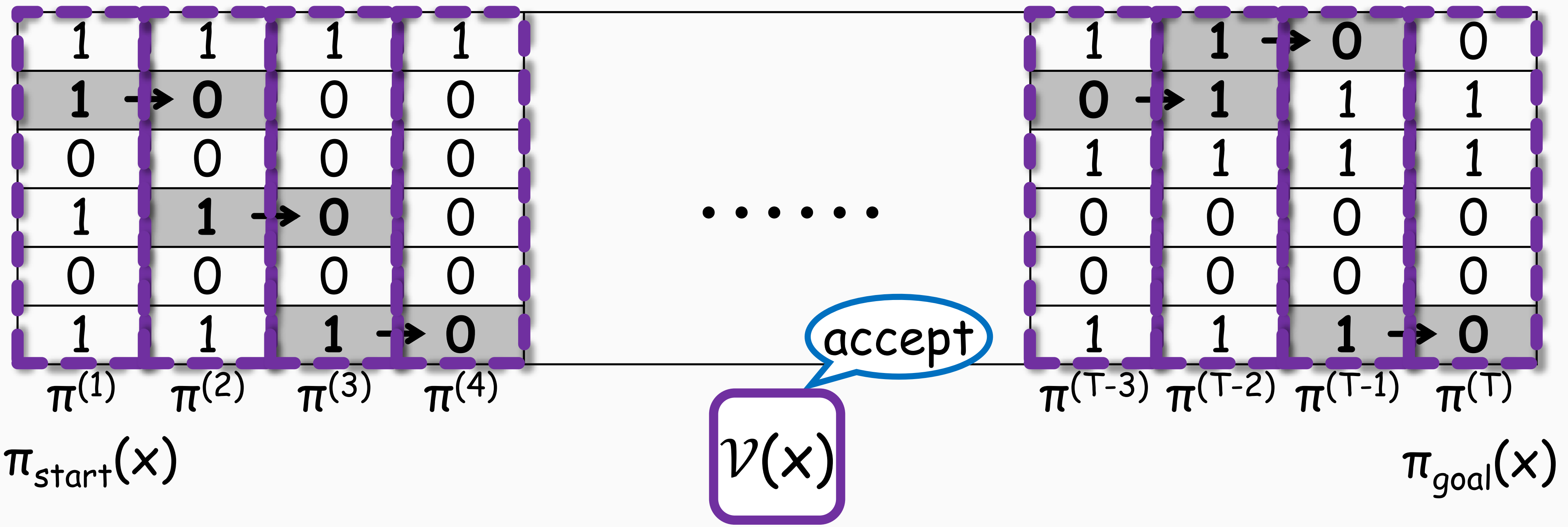
Probabilistically Checkable Reconfiguration Proofs [Hirahara-O. STOC 2024]

- Verifier V & poly-time alg. π_{start} & π_{goal} for language $L \subseteq \{0,1\}^*$

(Completeness)

$x \in L \implies \exists\, \boldsymbol{\pi} = (\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{start}(x)$ to $\pi_{goal}(x)$ s.t.

$$\forall t \;\; \Pr[\mathcal{V}(x) \text{ accepts } \pi^{(t)}] = 1$$



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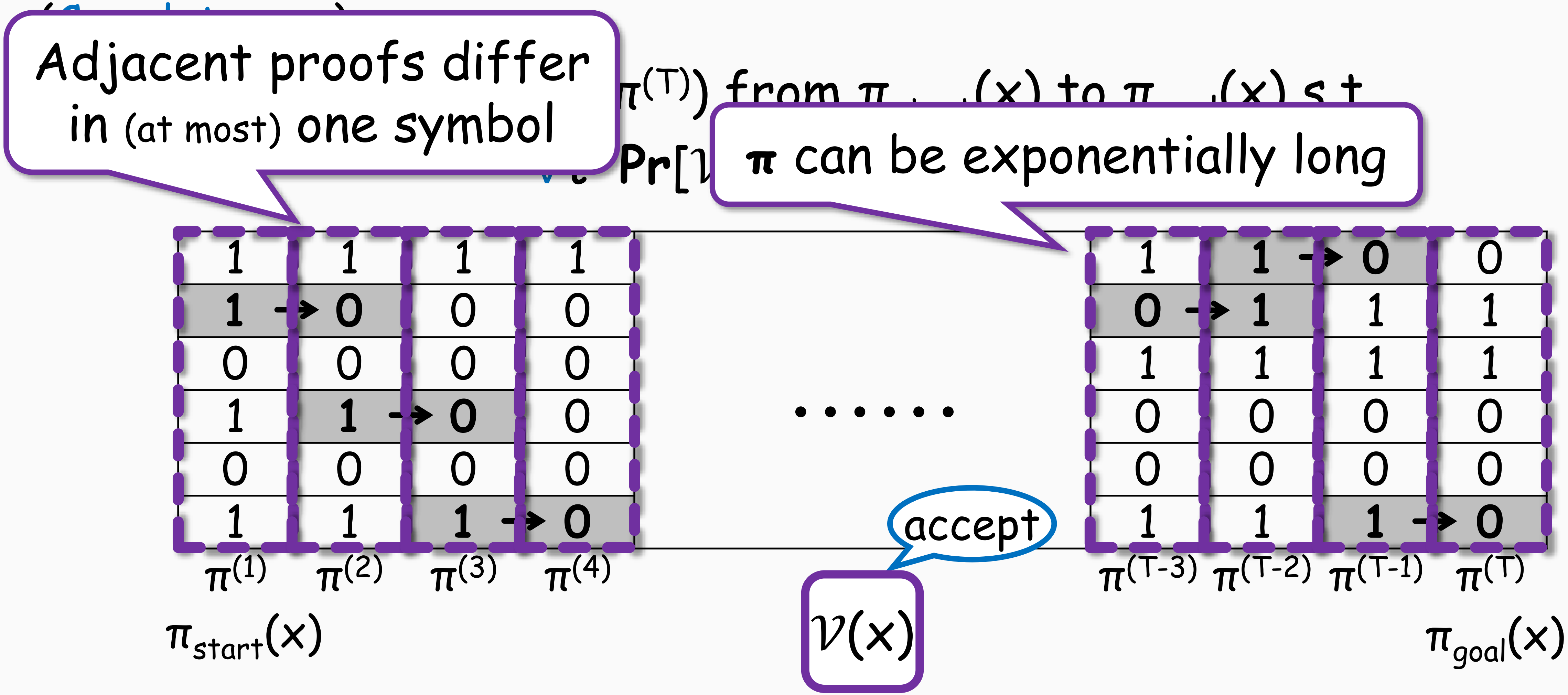


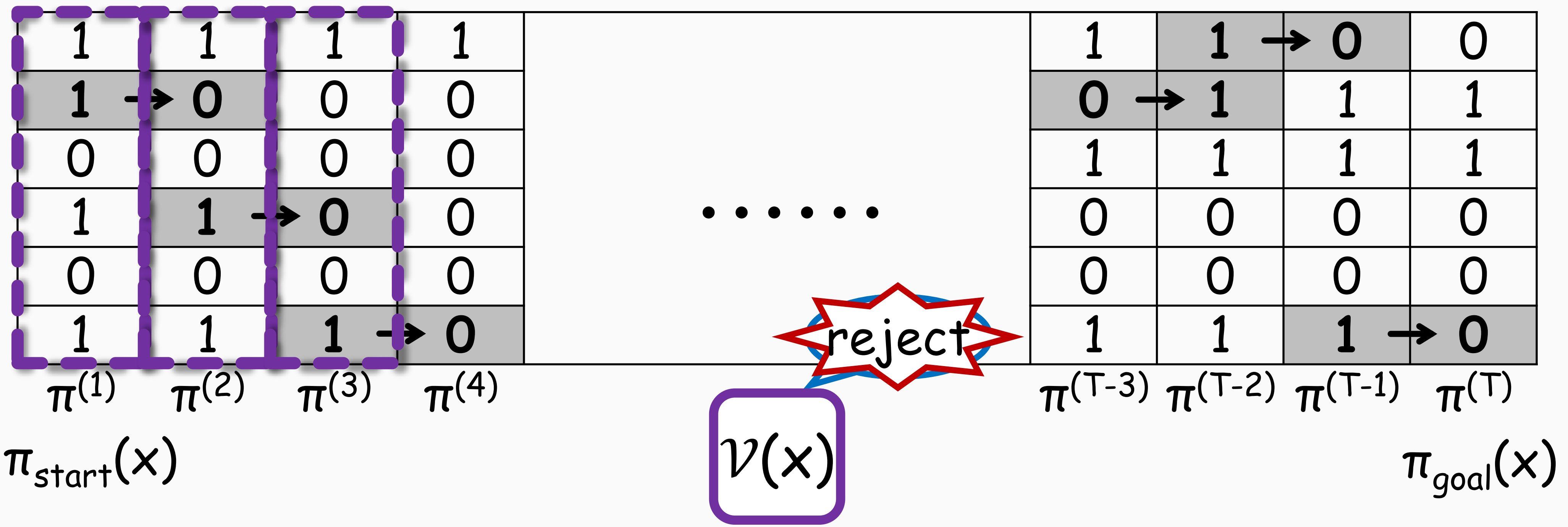
Probabilistically Checkable Reconfiguration Proofs [Hirahara-O. STOC 2024]

- Verifier V & poly-time alg. π_{start} & π_{goal} for language $L \subseteq \{0,1\}^*$

(Soundness)

$x \notin L \implies \forall \boldsymbol{\pi} = (\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{start}(x)$ to $\pi_{goal}(x)$,

$\exists t \;\; \Pr[\mathcal{V}(x) \text{ accepts } \pi^{(t)}] < \frac{1}{2}$



PCRP theorem [Hirahara-O. STOC 2024]

$$\mathbf{PSPACE} = \mathbf{PCRP}[O(\log n), O(1)]$$

$$L \in \mathbf{PSPACE}$$

$$\Updownarrow$$

$\exists$ Verifier $\mathcal{V}$ with randomness comp. $O(\log n)$ & query comp. $O(1)$

$\exists$ Poly-time alg. π_{start} & π_{goal}

Completeness $= 1$

Soundness $< \frac{1}{2}$

Proof sketch

🔁 Recap of verifier

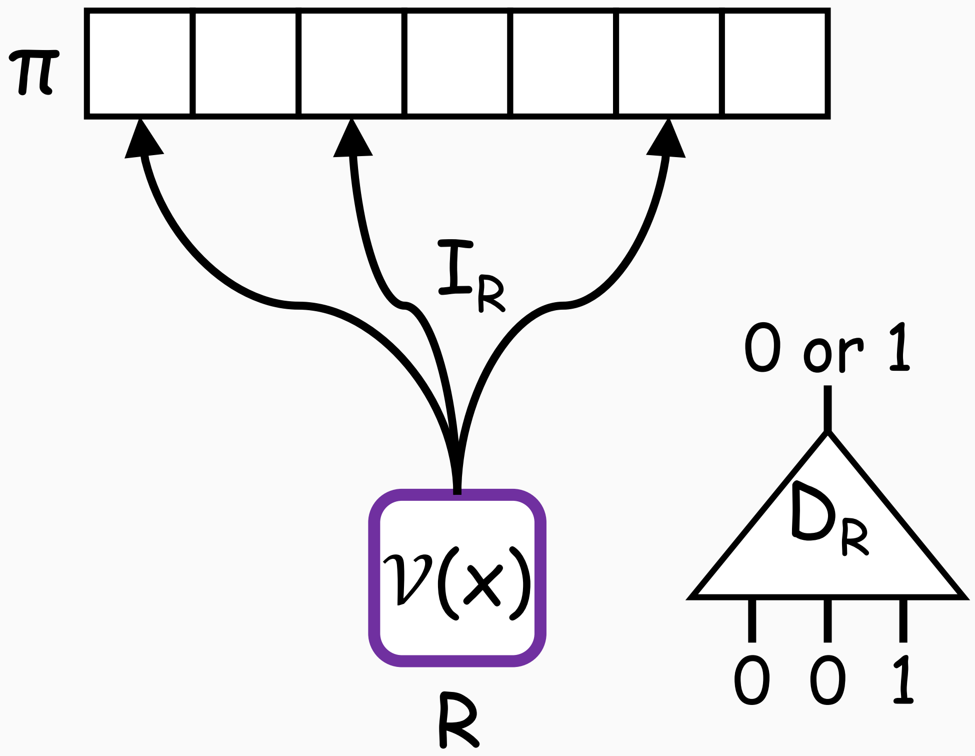
Verifier $\mathcal{V}$

Given: input $x \in \{0,1\}^n$

proof $\pi \in \{0,1\}^{poly(n)}$

- **1**. Sample random bits $R \in \{0,1\}^{r(n)}$
- **2**. Generate query seq. $I_R = (i_1, \ldots, i_{q(n)})$

  circuit $D_R: \{0,1\}^{q(n)} \rightarrow \{0,1\}$
- **3**. Accept iff $D_R(\pi|_{I_R})=1$

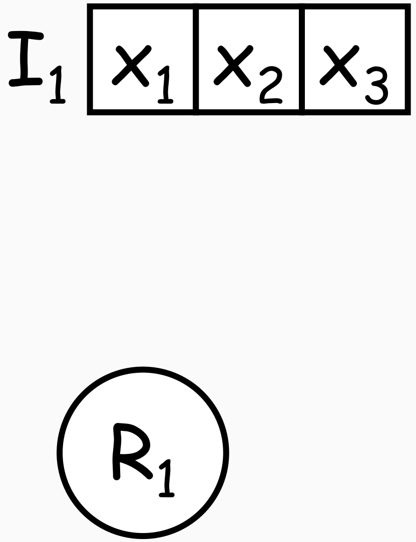


Proof sketch

Recap of FGLSS reduction

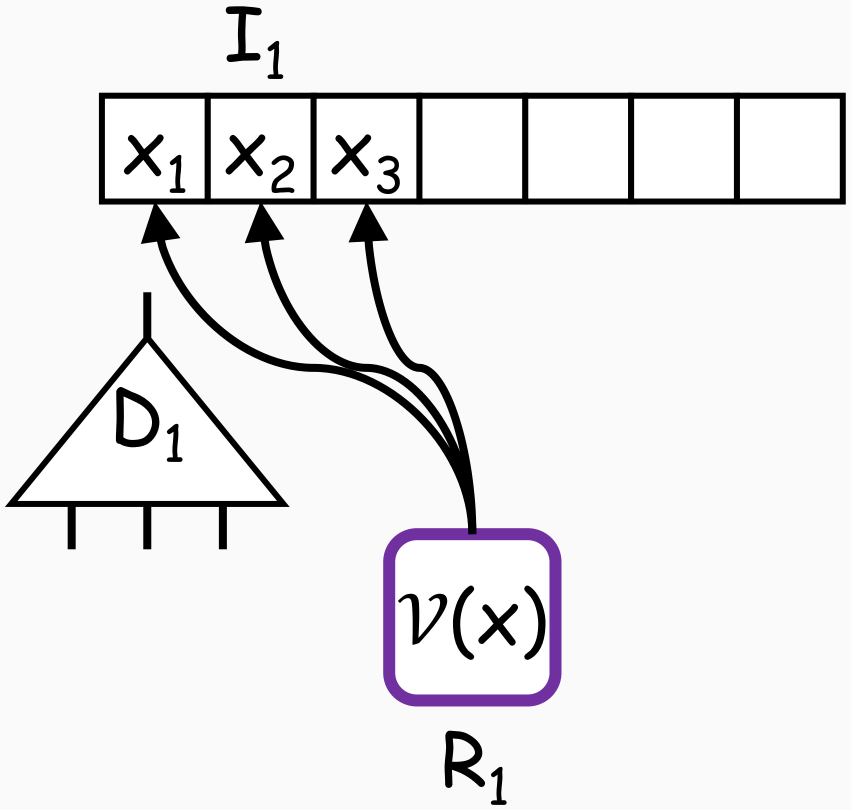
[Feige-Goldwasser-Lovász-Safra-Szegedy. J. ACM 1996]

CSP view



- $V := \{0,1\}^{r(n)}$

Verifier's view



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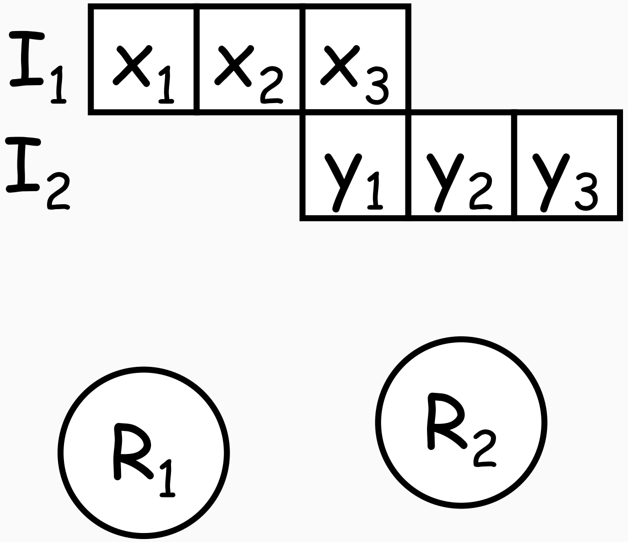

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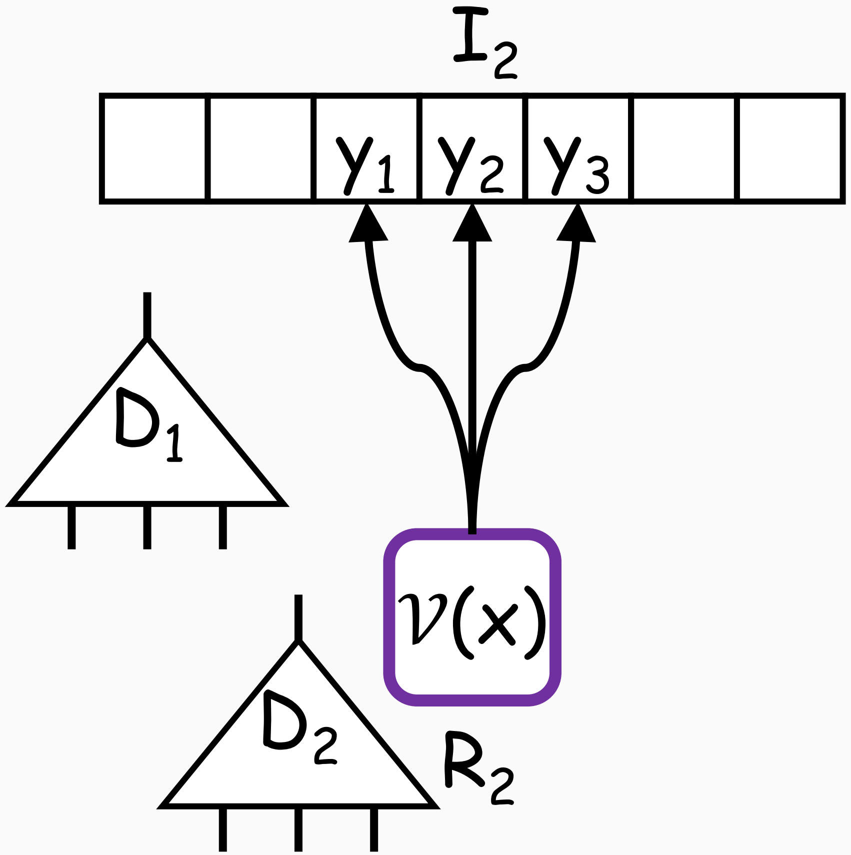

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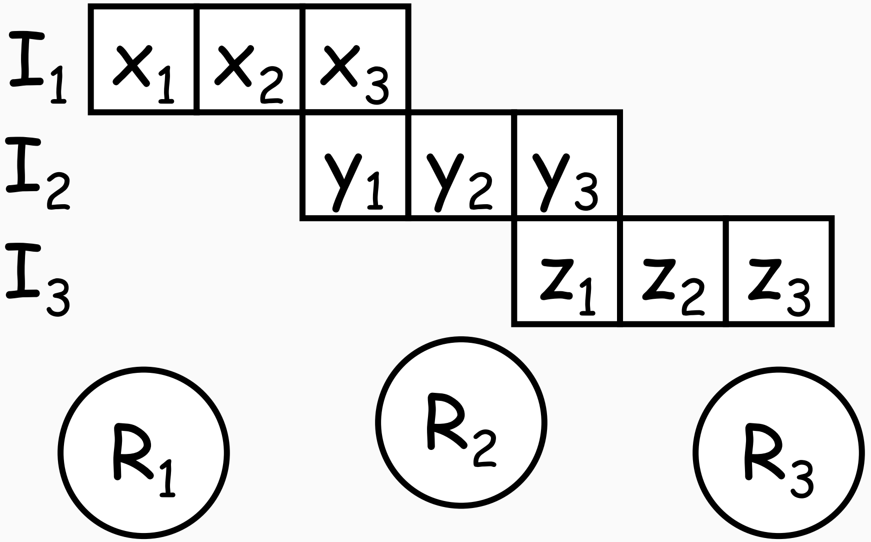

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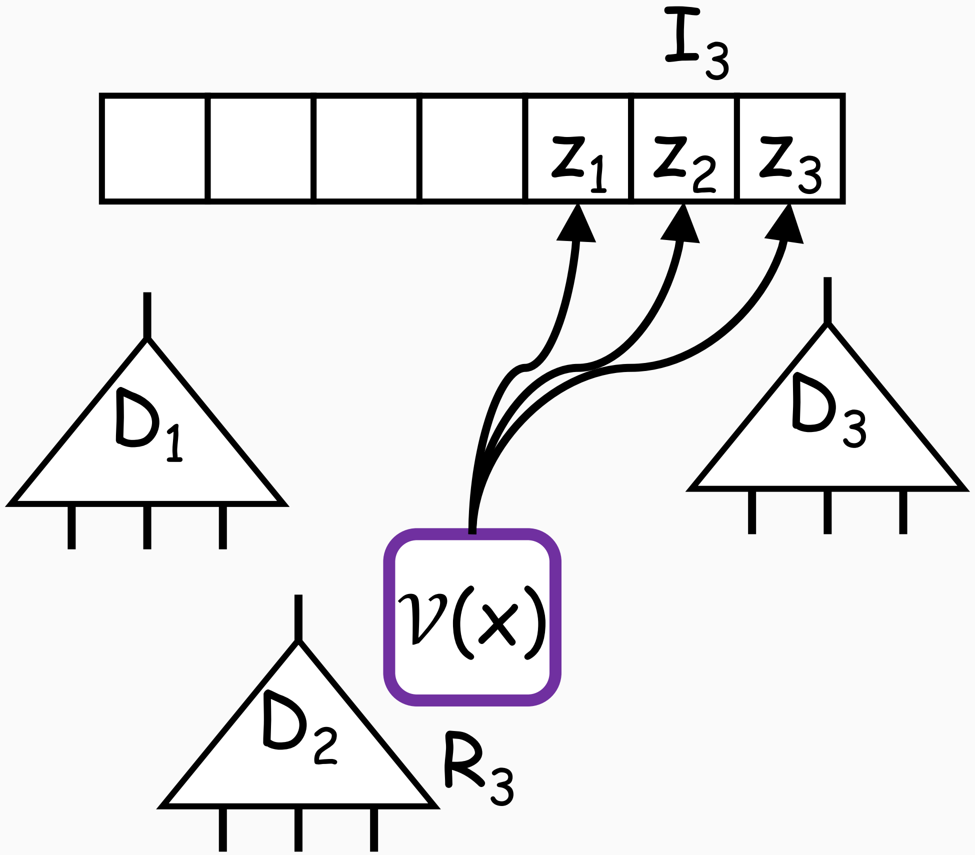

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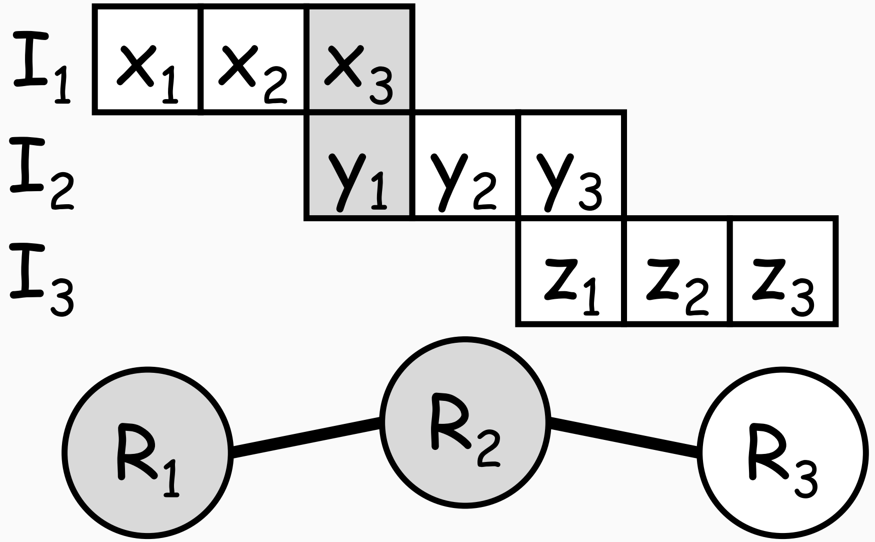
CSP view

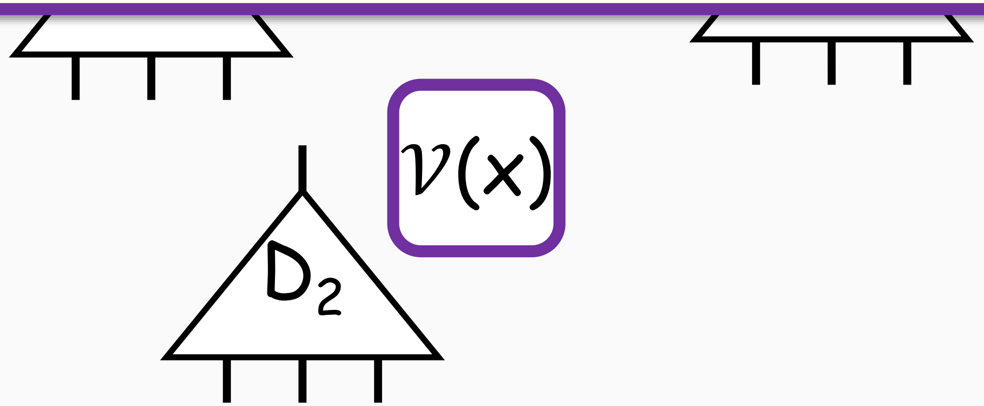


- $V := \{0,1\}^{r(n)}$
- $E := \{(R_i, R_j) : I_i \cap I_j \neq \emptyset\}$
- $\Sigma := \{0,1\}^{q(n)}$ (local view)
- $\Psi := (\psi_e)_{e \in E}$ where $\psi_e \colon \Sigma^e \to \{0,1\}$

$\boldsymbol{\alpha} \in \{0,1\}^{\{x_1,x_2,x_3\}}$ & $\boldsymbol{\beta} \in \{0,1\}^{\{y_1,y_2,y_3\}}$

$\psi_{R_1,R_2}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$ iff

- $D_1(\boldsymbol{\alpha}) = 1$ (feasibility)
- $D_2(\boldsymbol{\beta}) = 1$ (feasibility)
- $\boldsymbol{\alpha}_{x_3} = \boldsymbol{\beta}_{y_1}$ (consistency)



Proof sketch

Does FGLSS reduct. work for PCRP...?

CSP view | Verifier's view

$\exists f$ satisfies Ψ $\Longleftarrow$ $\exists \pi \; \mathbf{Pr}[\mathcal{V} \text{ accepts } \pi] = 1$

Completeness of 2-CSP | Completeness of PCP

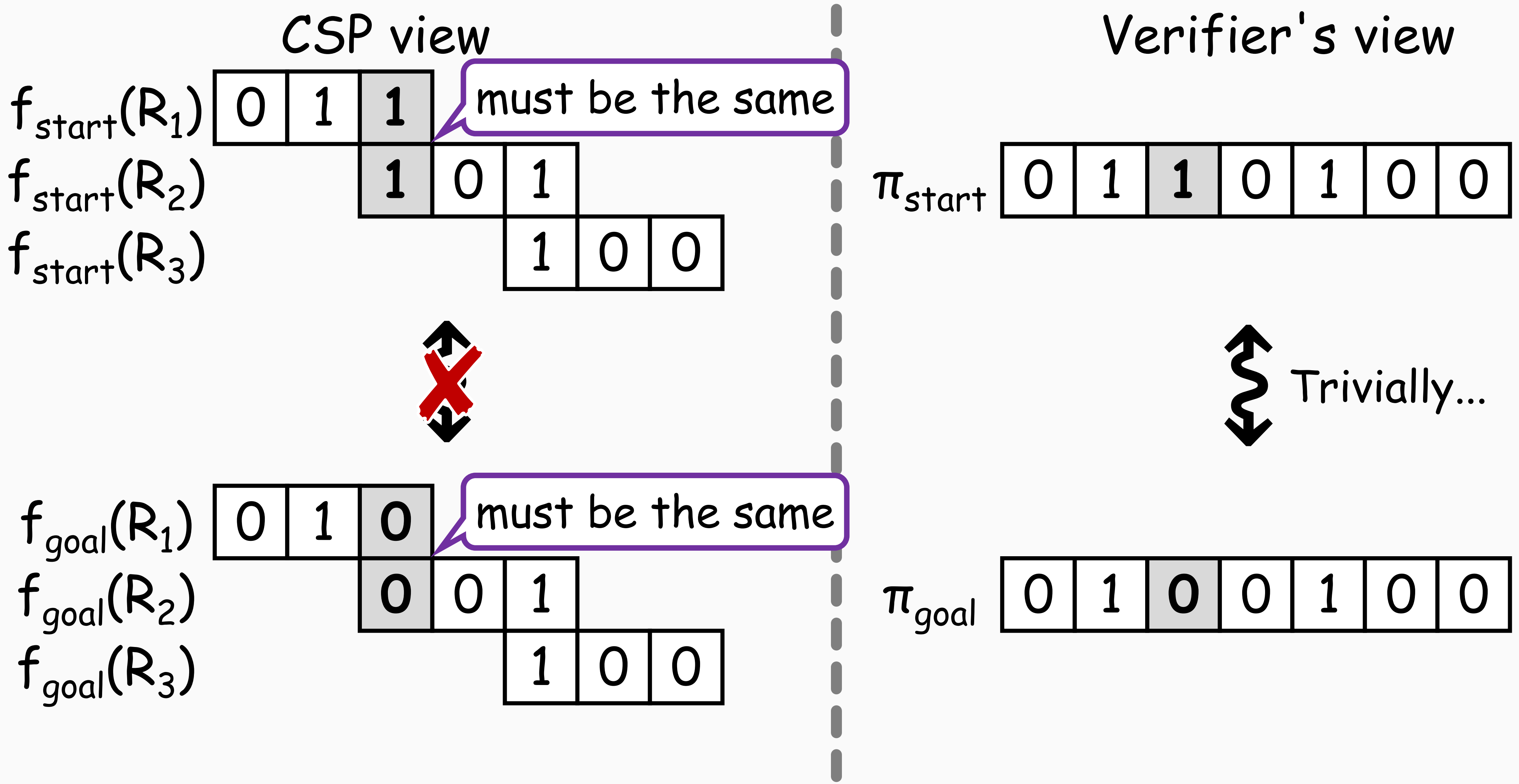
$\exists \mathbf{f} \; \forall f^{(t)}$ satisfies Ψ ✗$\Longleftarrow$ $\exists \boldsymbol{\pi} \; \forall \pi^{(t)} \; \mathbf{Pr}[\mathcal{V} \text{ acc. } \pi^{(t)}] = 1$

Completeness of 2-CSP Reconf. | Completeness of PCRP

😜 WHY...!?

Proof sketch

😭 Losing perfect completeness



Proof sketch

Alphabet squaring trick [O. STACS 2023 & SODA 2024]

🎯Think as if we could take a pair of values!

- Original $\Sigma = \{0, 1\}^{q(n)}$
- New $\Sigma_{sq} = \{0, 1, \mathbf{01}\}^{q(n)}$

	0	1	**01**
0	●		●
1		●	●
01	●	●	●

Intuition

- **01** takes 0 & 1 simultaneously
- x & y are **consistent** ⇔ x⊆y or x⊇y

😋Redefine ψ_e to "rescue" perfect completeness

(soundness analysis is nontrivial)

Conclusions

- 🔁Set Cover Reconf. is **PSPACE**-hard to approximate within 2-o(1)
- 👆FIRST sharp approx. threshold for reconf. problems
- 🔁Reconf. analogue of FGLSS reduction [Feige-Goldwasser-Lovász-Safra-Szegedy. J. ACM 1996] from PCRP [Hirahara-O. STOC 2024]
- More tight hardness of approx...?