2025.7.8 ICALP 2025 @ Aarhus, Denmark

Yet Another Simple Proof of the PCRP Theorem

Naoto Ohsaka

(CyberAgent Inc., Japan)

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Reconfiguration

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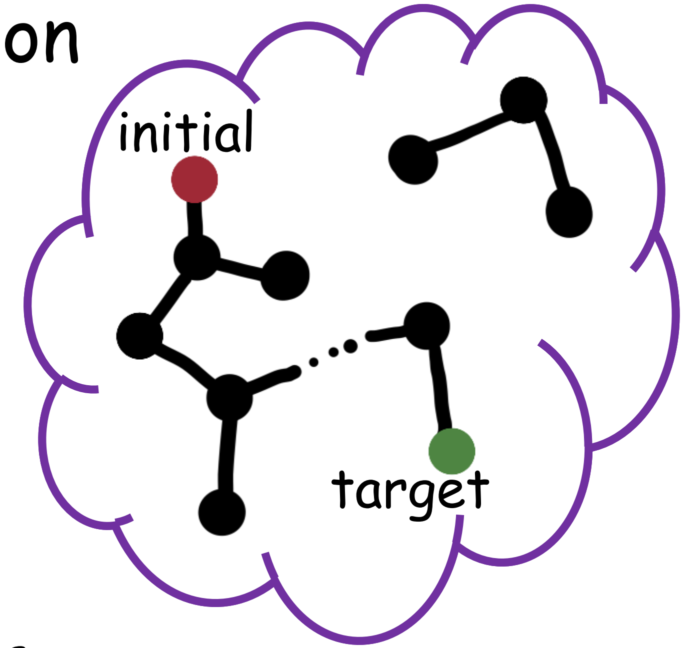
What is **reconfiguration**?

Imagine connecting a pair of feasible solutions (of NP problem)
under a particular adjacency relation



Q. Is a pair of solutions reachable to each other?

Q. If so, what is the shortest transformation?

Q. If not, how can the feasibility be relaxed?

Many reconfiguration problems have been derived from

SATISFIABILITY, COLORING, VERTEX COVER, CLIQUE, DOMINATING SET, FEEDBACK VERTEX SET, STEINER TREE, MATCHING, SPANNING TREE, SHORTEST PATH, SET COVER, SUBSET SUM, ...

See also [Nishimura 2018] [van den Heuvel 2013] [Hoang https://reconf.wikidot.com/]

Example 1

3-SAT RECONFIGURATION

[Gopalan/Kolaitis/Maneva/Papadimitriou 2009]

- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)} := \sigma_{ini}, \ldots, \sigma^{(T)} := \sigma_{tar})$ (reconf. sequence) s.t.

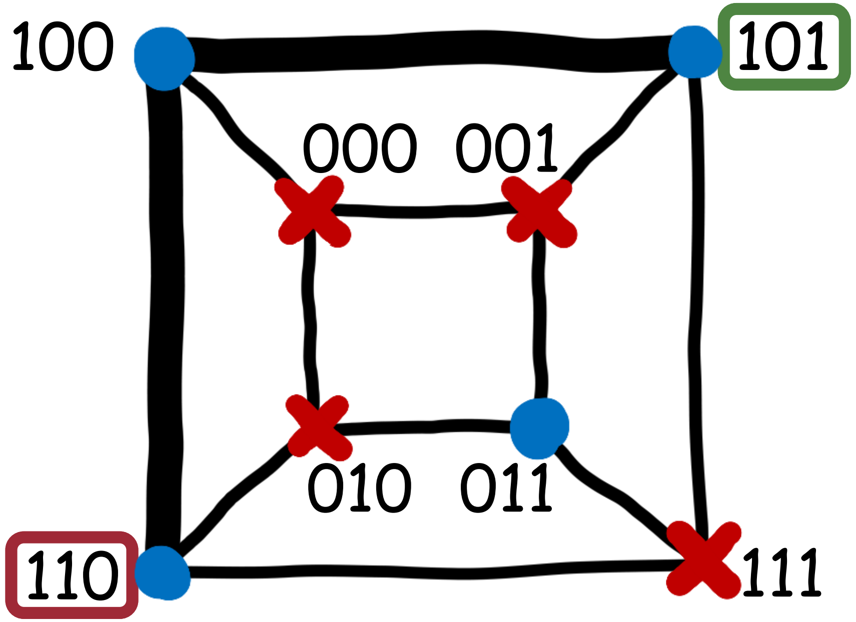
  $\sigma^{(t)}$ satisfies φ (feasibility)

  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)

$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,1,0)$

$\sigma_{tar}(x,y,z) = (1,0,1)$



YES

Example 2

3-SAT RECONFIGURATION

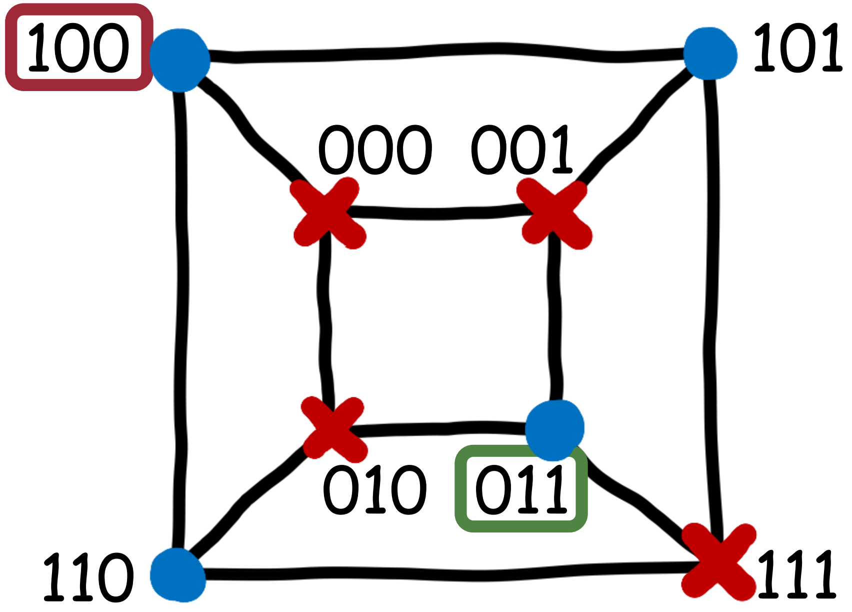
[Gopalan/Kolaitis/Maneva/Papadimitriou 2009]

- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)} := \sigma_{ini}, \ldots, \sigma^{(T)} := \sigma_{tar})$ (reconf. sequence) s.t.
  $\sigma^{(t)}$ satisfies φ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)

$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,0,0)$

$\sigma_{tar}(x,y,z) = (0,1,1)$



NO

Example 3

MAXMIN 3-SAT RECONFIGURATION

approximate version

[Ito/Demaine/Harvey/Papadimitriou/Sideri/Uehara/Uno 2011]

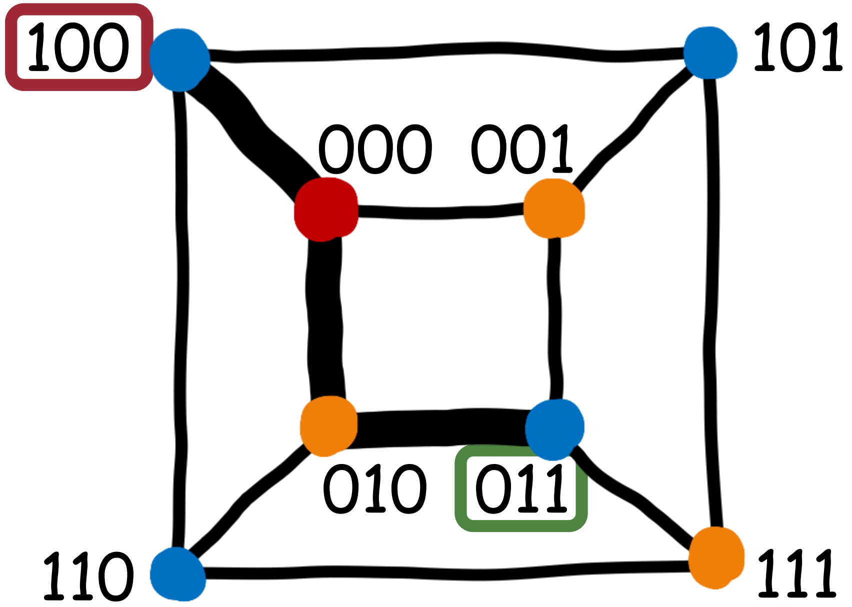
- **Input**: 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output**: $\vec{\sigma} = (\sigma^{(1)} := \sigma_{ini}, \ldots, \sigma^{(T)} := \sigma_{tar})$ (reconf. sequence) s.t.
  ~~$\sigma^{(t)}$ satisfies φ~~ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)
- **Goal**: maximize $\mathrm{val}_\varphi(\vec{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)

$$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

$$\sigma_{ini}(x,y,z) = (1,0,0)$$

$$\sigma_{tar}(x,y,z) = (0,1,1)$$

$$\mathrm{val}_\varphi(\vec{\sigma}) = \min\{1, \tfrac{1}{3}, \tfrac{2}{3}, 1\} = \tfrac{1}{3}$$



0.33

Example 4

MAXMIN 3-SAT RECONFIGURATION

approximate version

[Ito/Demaine/Harvey/Papadimitriou/Sideri/Uehara/Uno 2011]

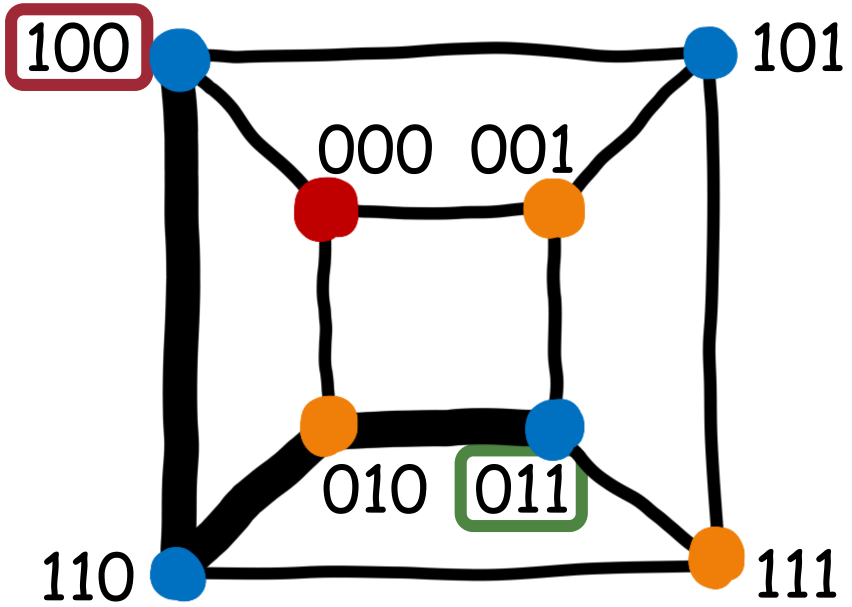
- **Input:** 3-CNF formula φ & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** $\vec{\sigma} = (\sigma^{(1)} := \sigma_{ini}, \ldots, \sigma^{(T)} := \sigma_{tar})$ (reconf. sequence) s.t.
  - ~~$\sigma^{(t)}$ satisfies φ~~ (feasibility)
  - $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)
- **Goal:** maximize $\mathrm{val}_{\varphi}(\vec{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)

$\varphi = (x \vee y) \wedge (x \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

$\sigma_{ini}(x,y,z) = (1,0,0)$

$\sigma_{tar}(x,y,z) = (0,1,1)$

$\mathrm{val}_{\varphi}(\vec{\sigma}) = \min\{1, 1, \frac{2}{3}, 1\} = \frac{2}{3}$



0.66

Our focus

Inapproximability of reconfiguration problems

- MAXMIN 3-SAT RECONFIGURATION is **PSPACE**-hard to solve exactly

  Since 3-SAT RECONFIGURATION is **PSPACE**-complete
  [Gopalan/Kolaitis/Maneva/Papadimitriou 2009]

- MAXMIN 3-SAT RECONFIGURATION is **NP**-hard to approximate

  Reduction from MAX 3-SAT
  [Ito/Demaine/Harvey/Papadimitriou/Sideri/Uehara/Uno 2011]

5. Open problems [Ito/Demaine/Harvey/Papadimitriou/Sideri/Uehara/Uno 2011]

There are many open problems raised by this work, and we mention some of these below:

- Can the MATCHING RECONFIGURATION problem for edge-weighted graphs be solved also in polynomial time? We conjecture that the answer is positive.
- Is the TRAVELING SALESMAN RECONFIGURATION problem (where two tours are adjacent if they differ in two edges) PSPACE-complete?
- Are there better approximation algorithms for the MINMAX POWER SUPPLY RECONFIGURATION problem? Lower bounds?
- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

🎉Probabilistically Checkable Reconfiguration Proofs

[Hirahara/O. STOC 2024]
[Karthik C. S./Manurangsi 2023]

Another PCP-type characterization of **PSPACE**
Probabilistically Checkable Debates [Condon/Feigenbaum/Lund/Shor 1995]

~~Reconfiguration~~ Probabilistically Checkable Proofs

[Arora/Lund/Motwani/Sudan/Szegedy 1998]
[Arora/Safra 1998]

∀ language L in **NP**

∃ verifier V with O(log n) randomness & O(1) query

(Completeness) $x \in L \Longrightarrow$

∃ proof π, $\mathbf{Pr}\left[V^{\pi}(x) = 1\right] = 1$

(Soundness) $x \notin L \Longrightarrow$

∀ proof π, $\mathbf{Pr}\left[V^{\pi}(x) = 1\right] < \frac{1}{2}$

🎉 Probabilistically Checkable Reconfiguration Proofs

[Hirahara/O. STOC 2024]
[Karthik C. S./Manurangsi 2023]

∀ language L in **PSPACE**

∃ verifier V with O(log n) randomness & O(1) query

∃ poly-time computable proofs π_{ini} & π_{tar}

(Completeness) $x \in L \implies$

∃ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{ini}(x)$ to $\pi_{tar}(x)$

$\forall\, t,\ \mathbf{Pr}\left[V^{\pi^{(t)}}(x) = 1\right] = 1$

(Soundness) $x \notin L \implies$

∀ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{ini}(x)$ to $\pi_{tar}(x)$

$\exists\, t,\ \mathbf{Pr}\left[V^{\pi^{(t)}}(x) = 1\right] < \frac{1}{2}$

Probabilistically Checkable Reconfiguration Proofs [

∀ language L in **PSPACE**

∃ verifier V with O(log n) randomnes

∃ poly-time computable proofs π_{ini} &

1	1	1	1	1	1
1	0	0	0	1	1
0	0	0	0	0	0
1	1	0	0	0	0
0	0	0	0	0	1
1	1	1	0	0	0
π_{ini}	$\pi^{(2)}$	$\pi^{(3)}$	$\pi^{(4)}$	$\pi^{(5)}$	π_{tar}

(Completeness) $x \in L \Longrightarrow$

∃ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{\text{ini}}(x)$ to $\pi_{\text{tar}}(x)$

$\forall\ t,\ \mathbf{Pr}\left[V^{\pi^{(t)}}(x) = 1\right] = 1$

(Soundness) $x \notin L \Longrightarrow$

∀ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{\text{ini}}(x)$ to $\pi_{\text{tar}}(x)$

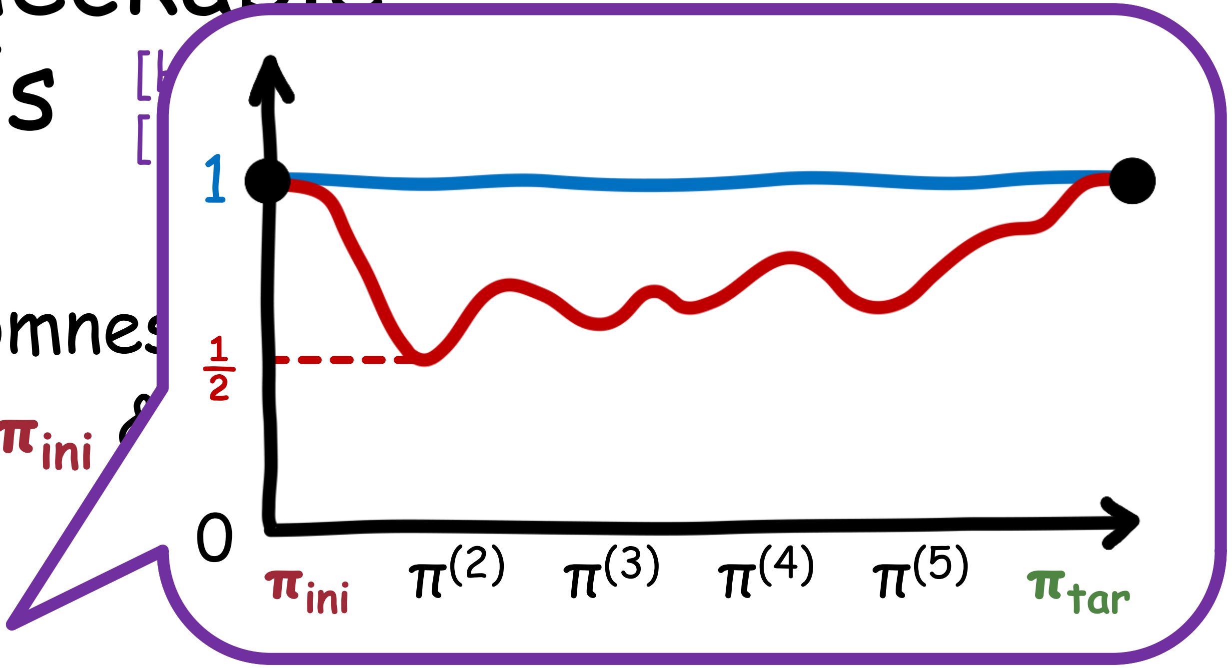
$\exists\ t,\ \mathbf{Pr}\left[V^{\pi^{(t)}}(x) = 1\right] < \frac{1}{2}$

Probabilistically Checkable Reconfiguration Proofs

∀ language L in **PSPACE**

∃ verifier V with O(log n) randomnes

∃ poly-time computable proofs π_{ini} &



(Completeness) $x \in L \Longrightarrow$

∃ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{\text{ini}}(x)$ to $\pi_{\text{tar}}(x)$

$\forall\, t,\ \Pr[V^{\pi^{(t)}}(x) = 1] = 1$

(Soundness) $x \notin L \Longrightarrow$

∀ reconf. sequence $(\pi^{(1)}, \ldots, \pi^{(T)})$ from $\pi_{\text{ini}}(x)$ to $\pi_{\text{tar}}(x)$

$\exists\, t,\ \Pr[V^{\pi^{(t)}}(x) = 1] < \frac{1}{2}$

Applications of the PCRP theorem

[Hirahara/O. STOC 2024] [Hirahara/O. ICALP 2024] [Hirahara/O. ICALP 2025]
[Karthik C. S./Manurangsi 2023] [O. STACS 2023] [O. SODA 2024] [O. ICALP 2024]

Reconfiguration problems of

3-SAT, 2-CSP, INDEPENDENT SET, VERTEX COVER,
CLIQUE, DOMINATING SET, SET COVER,
MAX CUT, and NONDETERMINISTIC CONSTRAINT LOGIC

are **PSPACE**-hard to approximate

within constant factor

Our contribution

Alternative proof of the PCRP theorem

"SIMPLER" than [Hirahara/O. STOC 2024] [Karthik C. S./Manurangsi 2023]

- **Robustization** [Ben-Sasson/Goldreich/Harsha/Sudan/Vadhan 2006] [Dinur/Reingold 2006]
- **Composition** [Arora/Lund/Motwani/Sudan/Szegedy 1998] [Arora/Safra 1998]

😋 Make construction more "modular" & analysis more "intuitive"

Proof overview

Robustization

Circuit Sat Reconfiguration

- **Input:** Boolean circuit C & satisfying asgmts. σ_{ini}, σ_{tar}
- **Output:** reconf. seq. $(\sigma^{(1)} := \sigma_{ini}, \ldots, \sigma^{(T)} := \sigma_{tar})$ s.t. $\forall t,\ C(\sigma^{(t)}) = 1$

🎯Given $(C, \sigma_{ini}, \sigma_{tar})$, we create $(\Phi, \pi_{ini}, \pi_{tar})$ s.t.

(Perfect completeness)

$(C, \sigma_{ini}, \sigma_{tar})$ is **YES** instance $\Longrightarrow$ $(\Phi, \pi_{ini}, \pi_{tar})$ is **YES** instance

(Robust soundness)

$(C, \sigma_{ini}, \sigma_{tar})$ is **NO** instance $\Longrightarrow$

$\forall$ reconf. sequence $(\pi^{(1)} := \pi_{ini}, \ldots, \pi^{(T)} := \pi_{tar})$

$\exists\ \pi^{(t)}$ that is 1%-far from $\Phi^{-1}(1)$ (= all satisfying asgmts.)

Proof overview

🔮NICE error-correcting code

$\exists \mathrm{Enc}: \{0,1\}^n \to \{0,1\}^{\mathrm{poly}(n)}$ s.t.

Concatenation of Reed-Solomon and Hadamard codes [David Forney 1965]

(**List decodability**) [Guruswami 2001]

Can list decode Enc relative radius $\frac{1}{3}$ in polynomial time

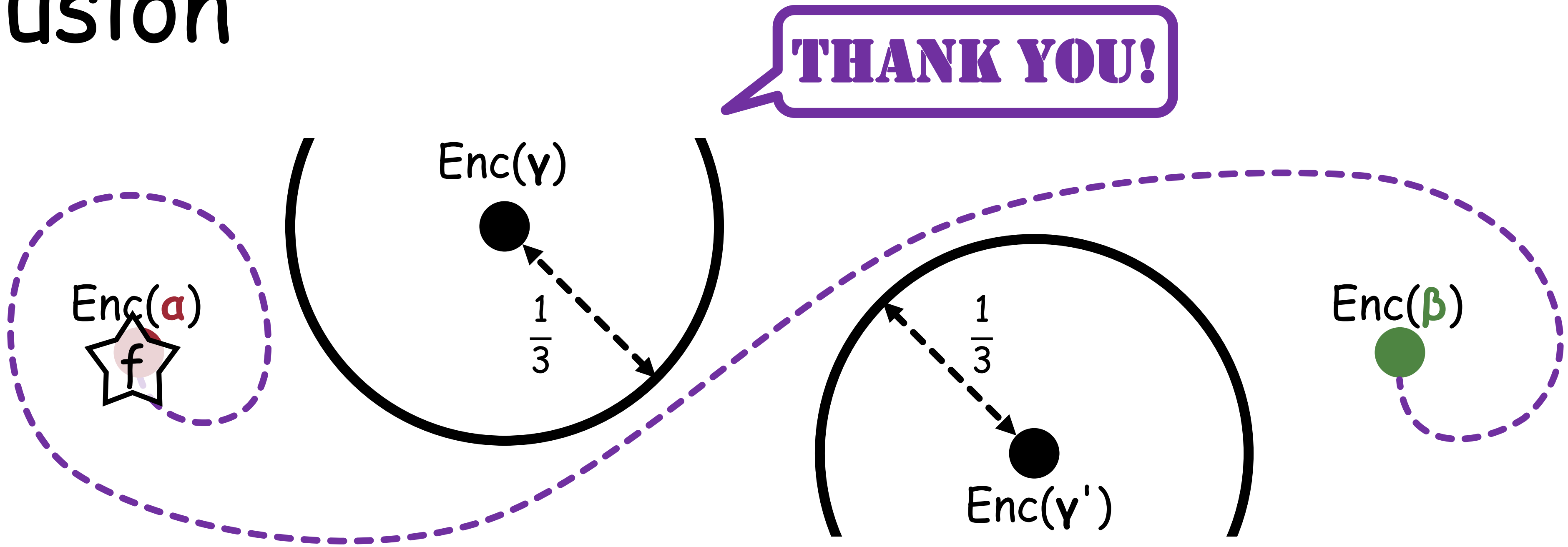
(**Reconfigurability**) [this paper] [O. ICALP 2024]

$\forall\ \alpha \neq \beta \in \{0,1\}^n$

$\exists$ reconf. sequence $(f^{(1)}, \ldots, f^{(T)})$ from $\mathrm{Enc}(\alpha)$ to $\mathrm{Enc}(\beta)$ s.t.

- $\delta(f^{(t)}, \mathrm{Enc}(\alpha)) \leq \frac{1}{4}$ or $\delta(f^{(t)}, \mathrm{Enc}(\beta)) \leq \frac{1}{4}$
- $\forall\ \gamma \neq \alpha, \beta \quad \delta(f^{(t)}, \mathrm{Enc}(\gamma)) > \frac{1}{3}$

Conclusion



🔁 Alternative proof of the PCRP theorem

- Other applications of the PCRP theorem?
- Other applications of "**reconfigurable**" error-correcting codes?