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Asymptotically Optimal Inapproximability of Maxmin k-Cut Reconfiguration

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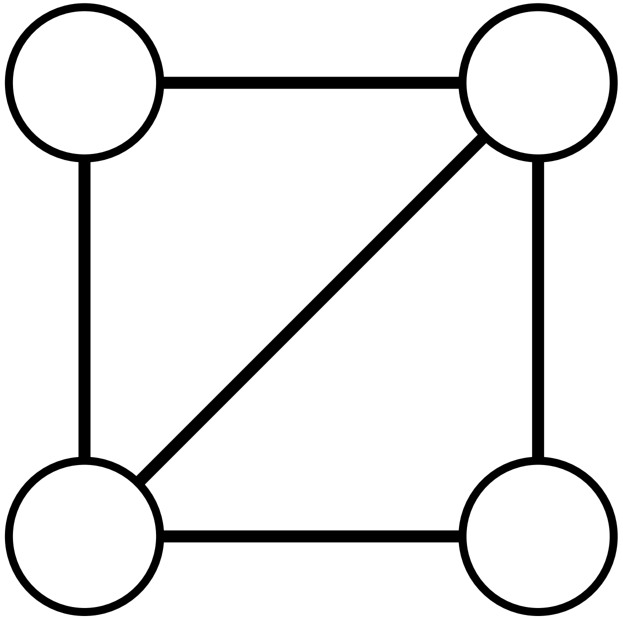
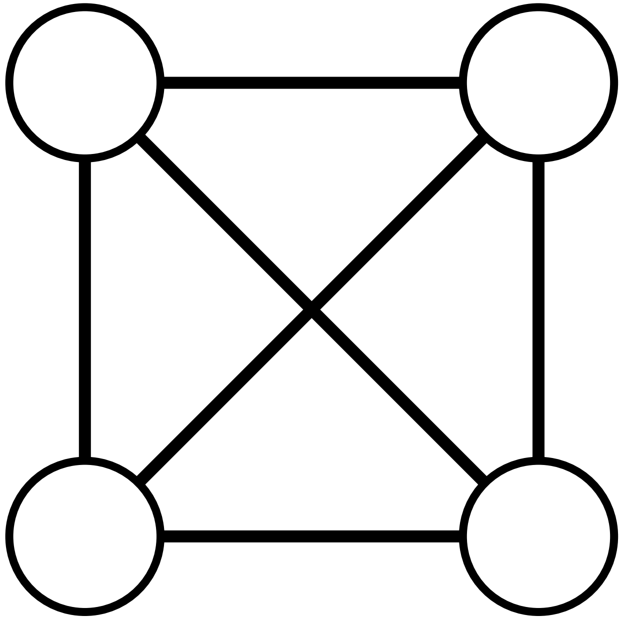
Example 1

k-Coloring

Q. Is there a proper 3-coloring of a given graph?





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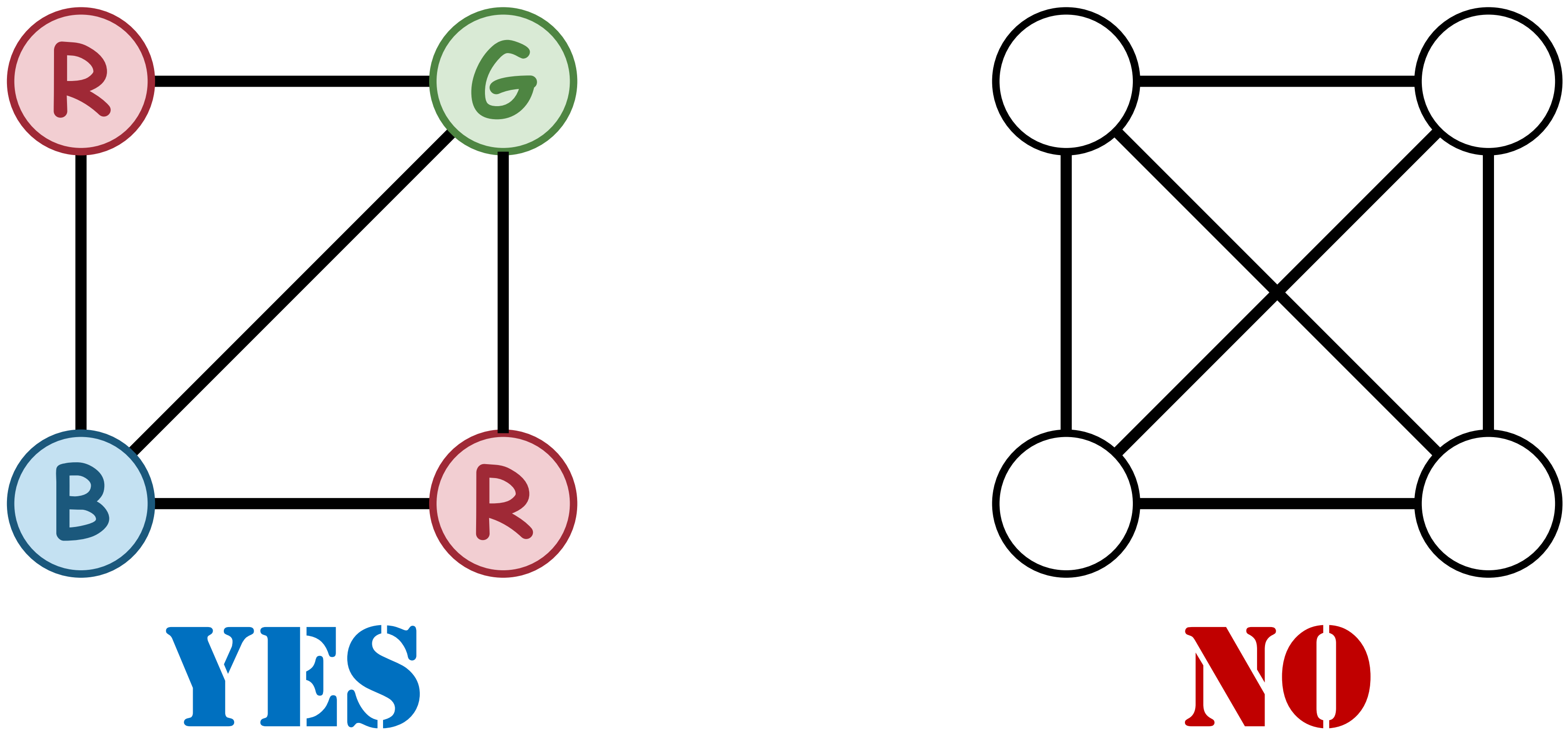
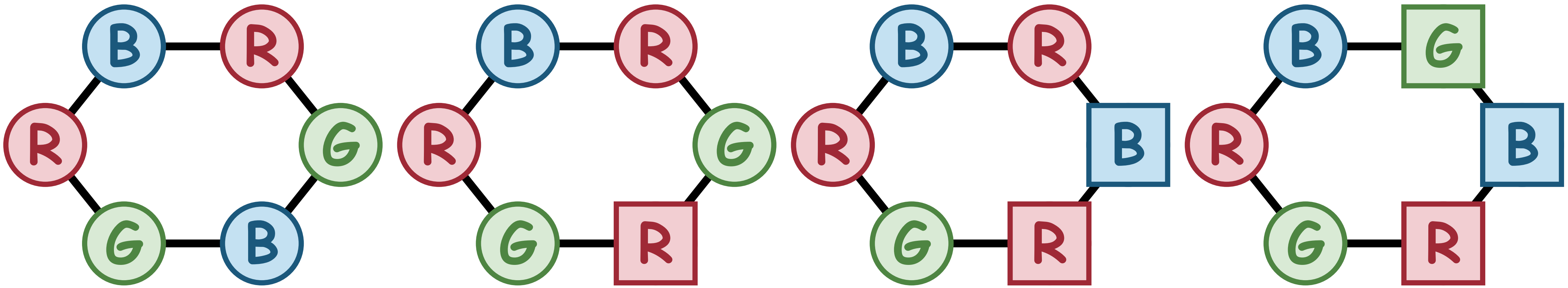


Example 2

k-Coloring Reconfiguration

[Cereceda–van den Heuvel–Johnson circa 2010]

- **Input:** graph G & proper k-colorings $\alpha_{ini}, \alpha_{tar}: V(G) \to [k]$
- **Output:** $\vec{\alpha} = (\alpha^{(1)} := \alpha_{ini}, \ldots, \alpha^{(T)} := \alpha_{tar})$ (reconf. sequence) s.t.
  - $\forall\ \alpha^{(t)}$ is proper (feasibility)
  - $\Delta(\alpha^{(t)}, \alpha^{(t+1)}) \leq 1$ (adjacency)



α_{ini}

YES

α_{tar}

Example 3

k-Coloring Reconfiguration

[Cereceda-van den Heuvel-Johnson circa 2010]

- **Input:** graph G & proper k-colorings $\alpha_{\mathrm{ini}}, \alpha_{\mathrm{tar}}: V(G) \to [k]$
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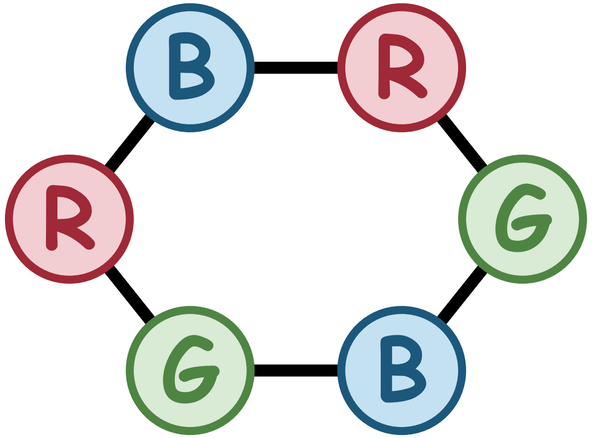
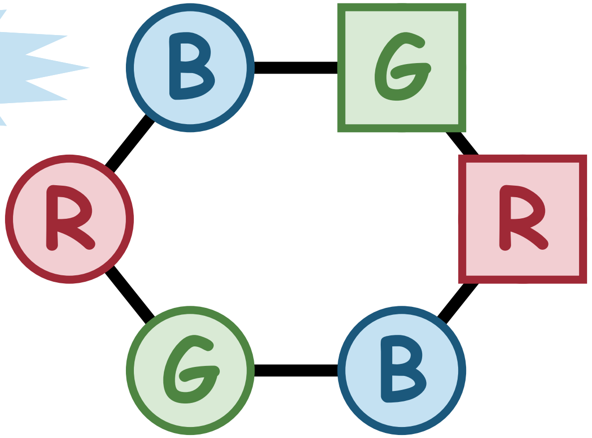


α_{ini}

frozen

NO



α_{tar}

Complexity of k-Coloring Reconfiguration

witness = coloring
polynomially long

k	2	3	4		Δ+1	≥ Δ+2
k-Coloring	**P**	**NP**-comp. [Garey–Johnson–Stockmeyer 1976] [Lovász 1973] [Stockmeyer 1973]			**YES** (greedy coloring)	
k-Coloring Reconf.	**P** [Cereceda–van den Heuvel 2011]		**PSPACE**-comp. [Bonsma–Cereceda 2009]			**YES** [Jerrum 1995]

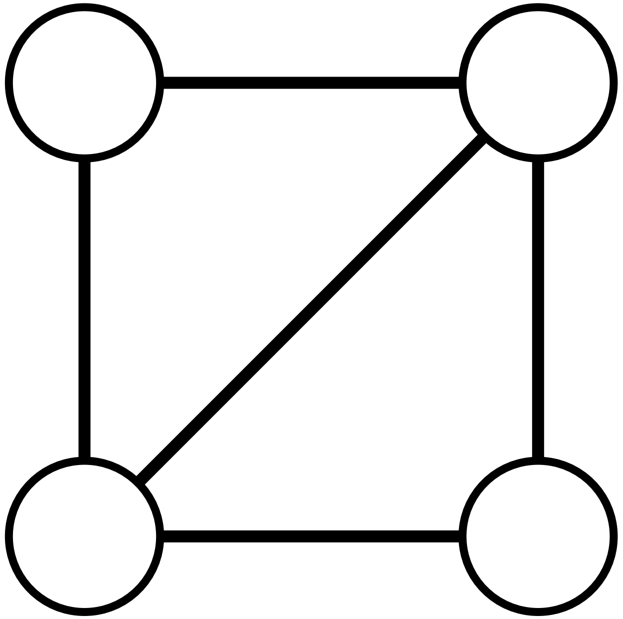
witness = reconf. sequence?
exponentially long

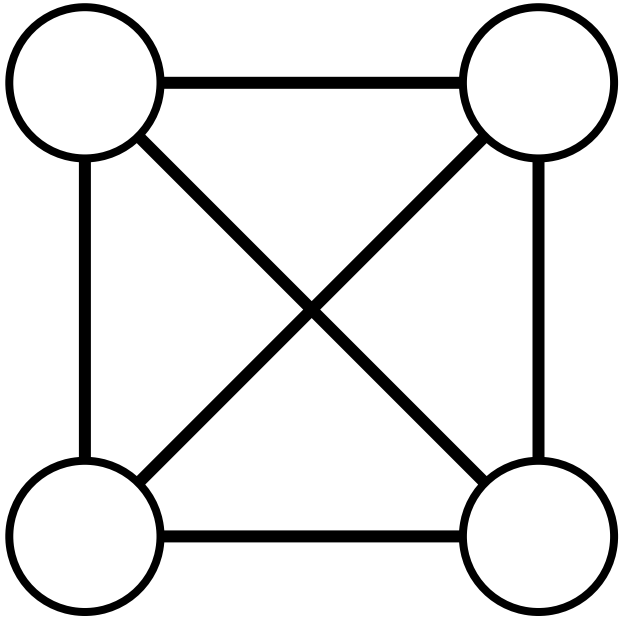
Example 4

Max k-Cut

approx. version of k-Coloring

Q. Find a 3-coloring maximizing frac. of bichromatic edges





Example 4

Max k-Cut

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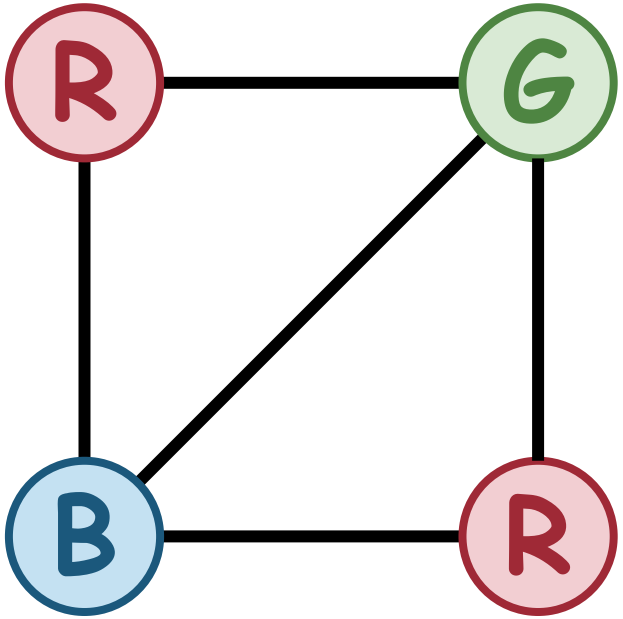

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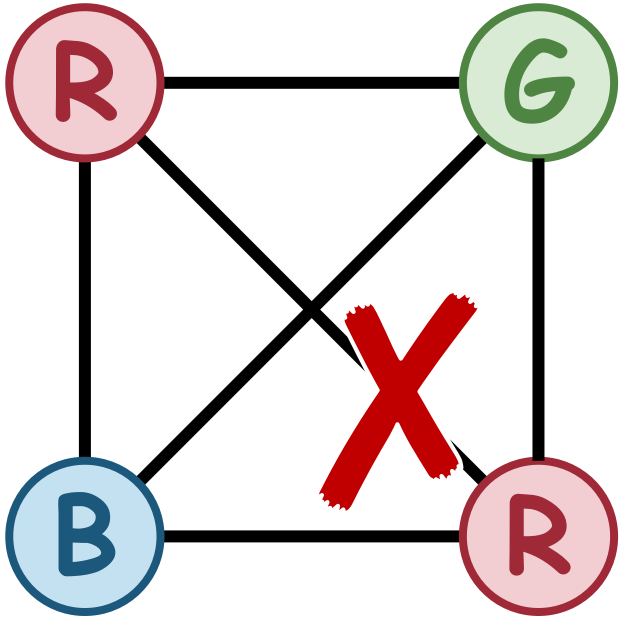

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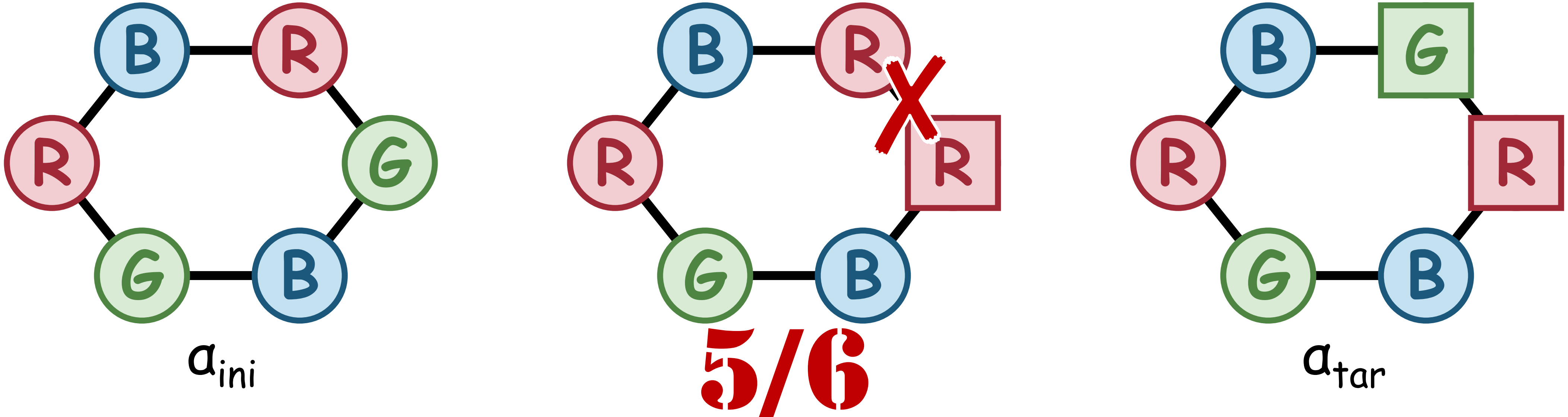
Example 5

Maxmin k-Cut Reconfiguration

approx. version of k-Coloring Reconf.

- **Input:** graph G & k-colorings $\alpha_{ini}, \alpha_{tar}: V(G) \to [k]$
- **Output:** $\vec{\alpha} = (\alpha^{(1)} := \alpha_{ini}, \ldots, \alpha^{(T)} := \alpha_{tar})$ (reconf. sequence) s.t.
  ~~$\forall \alpha^{(t)}$ is proper~~ (feasibility)
  $\Delta(\alpha^{(t)}, \alpha^{(t+1)}) \leq 1$ (adjacency)
- **Goal:** maximize $\min_t$ (frac. of bichromatic edges on $\alpha^{(t)}$)



Complexity of Maxmin k-Cut Reconf.

- **PSPACE**-hard to solve exactly $\forall\ k \geq 4$

  Since k-Coloring Reconfiguration is **PSPACE**-complete
  [Bonsma–Cereceda 2009]

- **PSPACE**-hard to approximate if $k = 4$

  Follows from the PCRP theorem + gap-preserving reductions
  [Bonsma–Cereceda 2009] [Hirahara–O. STOC 2024]
  [Karthik C. S.–Manurangsi 2023] [O. STACS 2023]

Q. What is asymptotic behavior of approximability w.r.t. the number k of available colors?

Our contribution

Optimal approx. factor $= 1 - \Theta(\frac{1}{k})$

- **PSPACE**-hardness of $(1 - \frac{\varepsilon}{k})$-approx.
- $(1 - \frac{2}{k})$-approx. algorithm

Just like Max k-Cut... not surprising.
[Kann-Khanna-Lagergren-Panconesi 1997] [Guruswami-Sinop 2013]

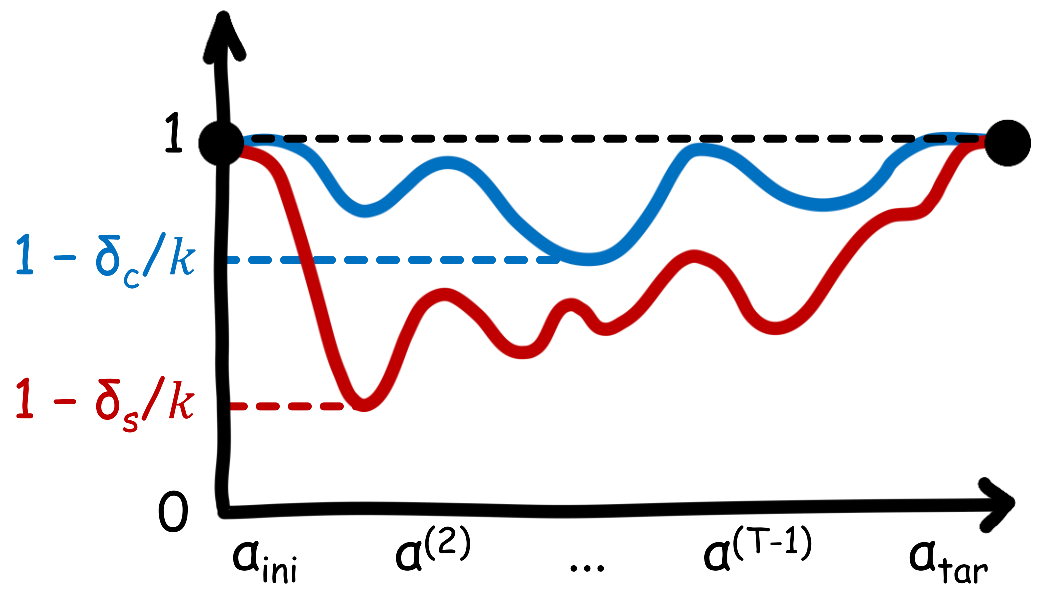
Gap reductions between reconfiguration problems are NONTRIVIAL

Proof overview

PSPACE-hardness

Input: graph G & k-colorings α_{ini}, α_{tar}

$\mathrm{opt}_G(\alpha_{ini}, \alpha_{tar}) :=$ opt. value



PSPACE-hard to distinguish between

(Completeness) $\mathrm{opt}_G(\alpha_{ini}, \alpha_{tar}) \geq 1 - \delta_c/k$

∃reconf. sequence ∀k-coloring $(1 - \delta_c/k)$-frac. of edges are bichromatic

(Soundness) $\mathrm{opt}_G(\alpha_{ini}, \alpha_{tar}) < 1 - \delta_s/k$

∀reconf. sequence ∃k-coloring (δ_s/k)-frac. of edges are monochromatic

∴ Maxmin k-Cut Reconfiguration is **PSPACE**-hard

to approximate within $1 - (\delta_s - \delta_c)/k$

Proof overview

🎯 Main gap-preserving reduction

Gap 2-Cut Reconf. $(G, \alpha_{ini}, \alpha_{tar})$

- $\mathrm{opt}_G(\alpha_{ini}, \alpha_{tar}) \geq 1 - \varepsilon_c$
- (Completeness)
- (Soundness)
- $\mathrm{opt}_G(\alpha_{ini}, \alpha_{tar}) < 1 - \varepsilon_s$

Gap k-Cut Reconf. $(H, \beta_{ini}, \beta_{tar})$

- $\mathrm{opt}_H(\beta_{ini}, \beta_{tar}) \geq 1 - \delta_c/k$
- (Completeness)
- (Soundness)
- $\mathrm{opt}_H(\beta_{ini}, \beta_{tar}) < 1 - \delta_s/k$

😋 $\delta_c < \delta_s$ depend only on $\varepsilon_c < \varepsilon_s$

- **PSPACE**-hardness of 👆 follows from the PCRP theorem (two talks ago!)
  [Bonsma-Cereceda 2009] [Hirahara-O. STOC 2024]
  [Karthik C. S.-Manurangsi 2023] [O. STACS 2023]

Proof overview

Why existing reductions do not work?

Apply reduction from Max 2-Cut to Max k-Cut
[Kann-Khanna-Lagergren-Panconesi 1997] [Guruswami-Sinop 2013]

2-coloring α_{ini} of G

2-coloring α_{tar} of G

⚠ $\text{opt}_G(\alpha_{\text{ini}}, \alpha_{\text{tar}}) = \frac{1}{2}$

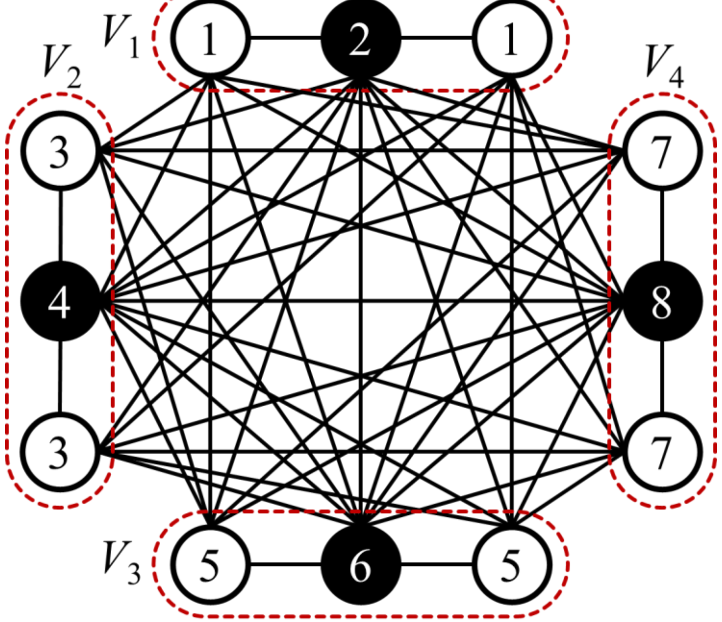
Proof overview

Why existing reductions do not work?

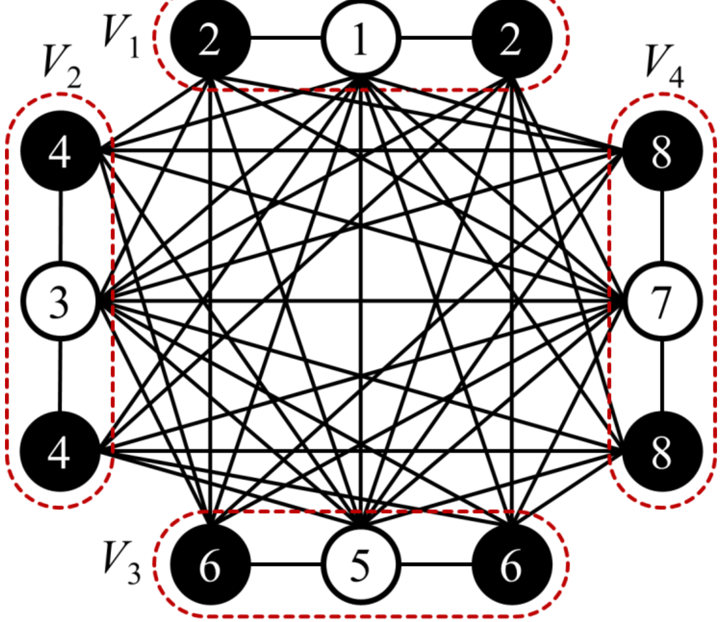
Apply reduction from Max 2-Cut to Max k-Cut

[Kann-Khanna-Lagergren-Panconesi 1997] [Guruswami-Sinop 2013]

k-coloring β_{ini} of H



k-coloring β_{tar} of H



$\text{opt}_H(\beta_{ini}, \beta_{tar}) \geq 1 - O(1/k^2)$

Recolor vertices of $V_1, V_2, \ldots, V_{k/2}$ in this order

Proof overview

Our construction

💡 "Encode" a 2-coloring of a vertex by a k-coloring of a $k \times k$ grid

(See the paper for motivation)

2-coloring α_{ini} of G

2-coloring α_{tar} of G

Our construction

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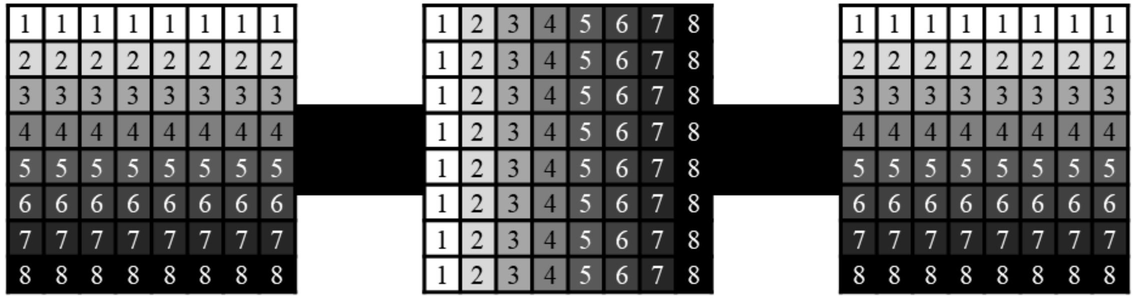
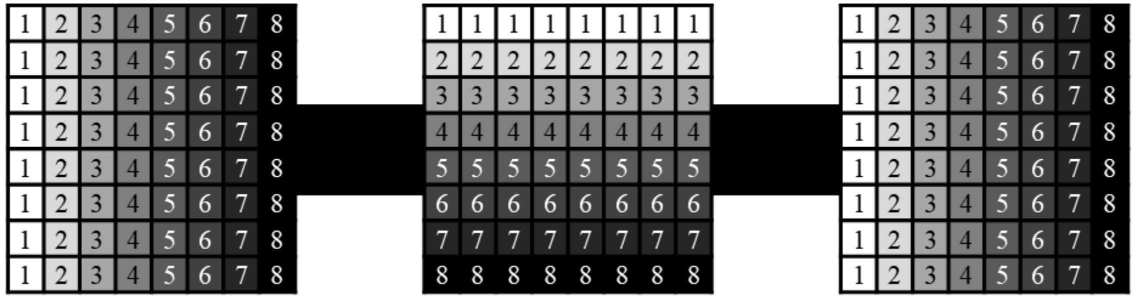


"horizontal stripe" = 1

k-coloring β_{tar} of H



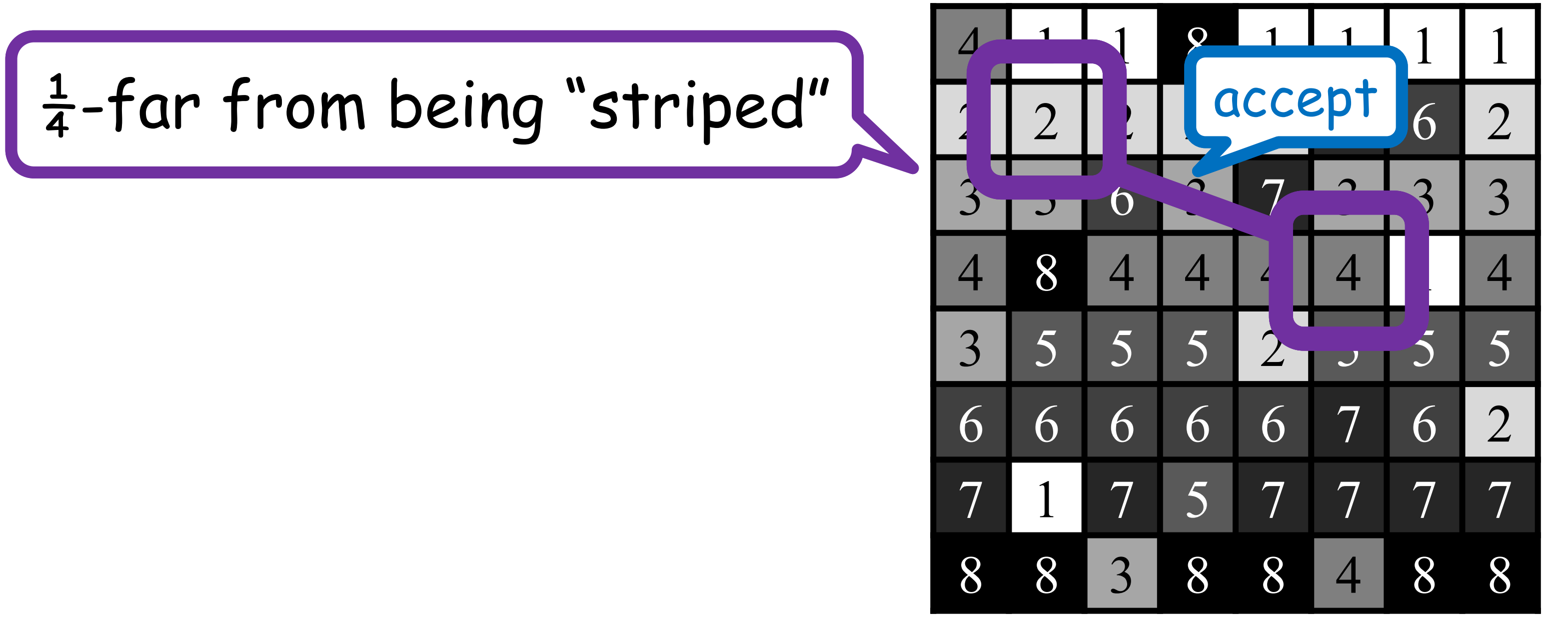
"vertical stripe" = 2

🎯 Test if a k-coloring of H "encodes" a (near-)proper 2-coloring of G

Proof overview

"Stripe" testing

Test if a k-coloring of a $k\times k$ grid is close to being "striped"



2-query verifier V given oracle access to f: $[k]^2 \to [k]$ s.t.

- f is "striped" $\implies$ V^f accepts w.p. 1
- f is ε-far from being "striped" $\implies$ V^f rejects w.p. $\Omega(\varepsilon/k)$

Proof overview

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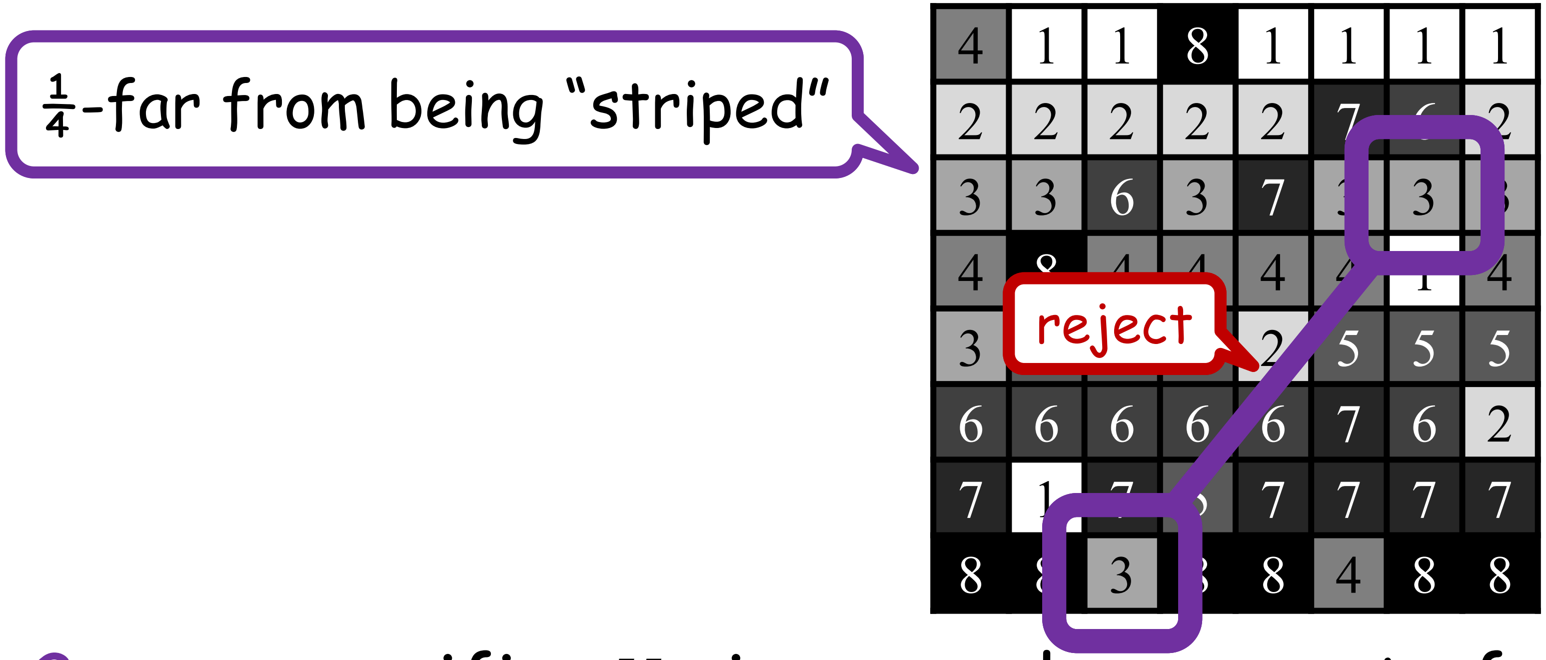

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Conclusion

🔁 Approx. threshold of Maxmin k-Cut Reconf. $= 1 - \Theta(1/k)$

- Optimal hidden constant in $\Theta(1/k)$?
- Perfect completeness?
- Improved analysis of (generalized) "stripe" testing?

THANK YOU!

4	1	1	8	1	1	1	1
2	2	2	2	2	7	6	2
3	3	6	3	7	3	3	3
4	8	4	4	4	4	1	4
3	5	5	5	2	5	5	5
6	6	6	6	6	7	6	2
7	1	7	5	7	7	7	7
8	8	3	8	8	4	8	8