ERATO Kawarabayashi Large Graph Project

Monotone k-Submodular Function Maximization with Size Constraints

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Introduction

Task: select a set $S \subseteq V$ ($|V| = n$) of items of a specific size
Goal: maximize a monotone ***submodular*** set function $f: 2^V \to \mathbb{R}$

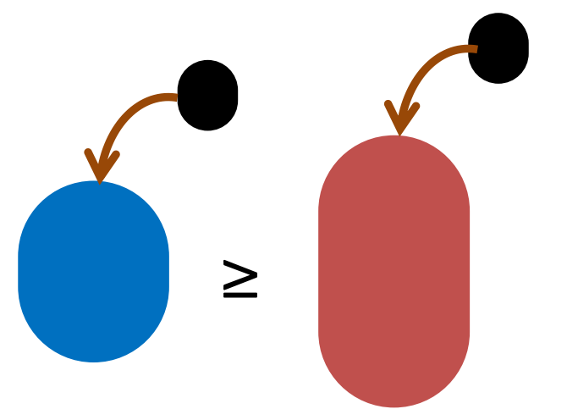
$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$$

for any $S, T \subseteq V$

Equivalent to the ***diminishing return*** property

$$f(S+e) - f(S) \geq f(T+e) - f(T)$$

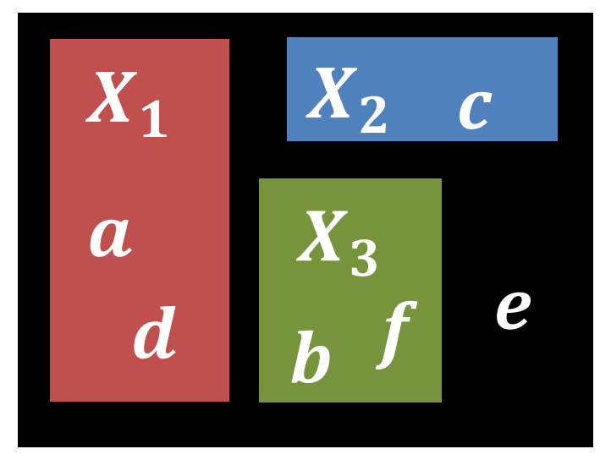
for any $S \subseteq T$ and $e \in V \setminus T$



Greedy strategy
for submodular maximization
Simple & Fast
Approx. ratio $= 1 - e^{-1} \approx 0.63$
[Nemhauser-Wolsey-Fisher. Math. Program. '78]

Various applications
sensor placement
influence maximization
document summarization
feature selection
network inferring …

More complex situations



e	a	b	c	d	e	f
$x(e)$	1	3	2	1	0	3

What if selecting k disjoint sets? What if assigning k kinds of items?

We adopt ***k*-submodular** functions

Our contributions

Approximation algorithms

For monotone k-submodular function maximization
under size constraints

① Total size constraint
Given a total budget B
for k kinds of items
$\# ● + \# ★ + \# ◆ = B$
Approx. ratio $= 1/2$
\# function eval. $=$
$\mathcal{O}(knB)$ Greedy strategy
$\tilde{O}(kn)$ Random sampling

② Individual size constraint
Given a budget B_i
for each kind of items
$\# ● = B_1, \# ★ = B_2, \# ◆ = B_3$
Approx. ratio $= 1/3$
\# function eval. $=$
$\mathcal{O}(knB)$ Greedy strategy
$\tilde{O}(k^2 n)$ Random sampling

Experimental evaluations

Influence maximization with k topics
Sensor placement with k kinds of measures

Related work

Theoretical results under **NO** constraint [Iwata-Tanigawa-Yoshida. SODA'16]

$1/2$-approx. algorithm for non-monotone k-submodular maximization
$\frac{k}{2k-1}$-approx. algorithm for monotone k-submodular maximization

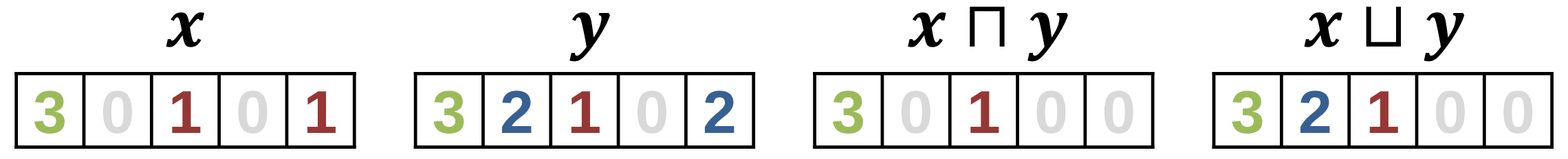
Applications of bi(2-)submodular functions [Singh-Guillory-Bilmes. AISTATS'12]

Sensor placement & feature selection
No approx. guarantee

k-submodular functions

$f: (k+1)^V \to \mathbb{R}$ is ***k-submodular*** if for any $\boldsymbol{x}$ & $\boldsymbol{y}$

$$f(\boldsymbol{x}) + f(\boldsymbol{y}) \geq f(\boldsymbol{x} \sqcap \boldsymbol{y}) + f(\boldsymbol{x} \sqcup \boldsymbol{y})$$



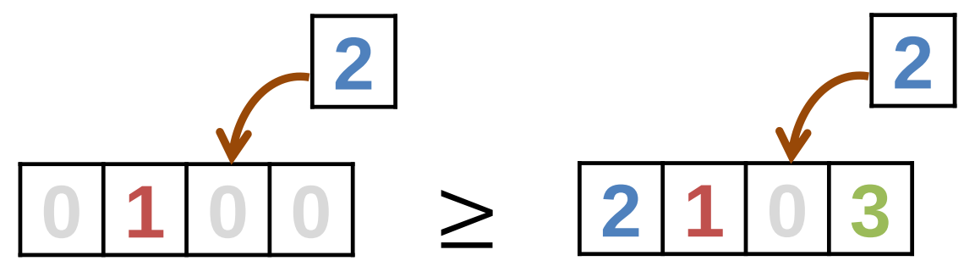
$\sqcap$	0	i	j
0	0	0	0
i	0	i	0
j	0	0	j

$\sqcup$	0	i	j
0	0	i	j
i	i	i	0
j	j	0	j

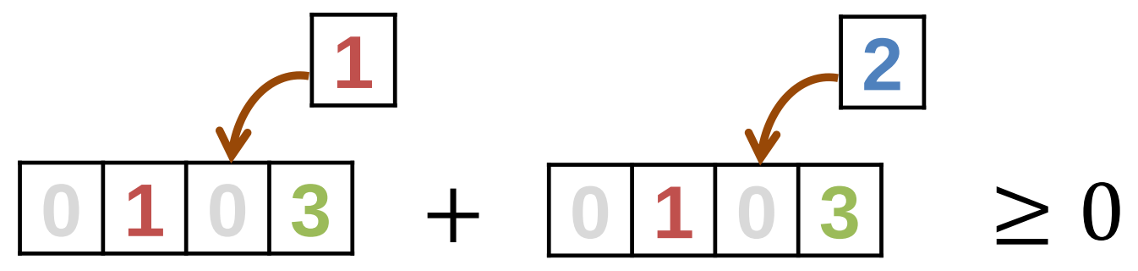
Characterization [Ward-Živný. ACM Trans. Algor. '15]

A function f is k-submodular *if and only if* f satisfies ① & ②

① Orthant submodular



② Pairwise monotone



② irrelevant in this work since we consider *monotone* functions

Total size constraint

Given: a monotone k-submodular function f & an integer $B \leq n$
Goal: max $f(\boldsymbol{x})$ s.t. $|\mathrm{supp}(\boldsymbol{x})| \leq B$

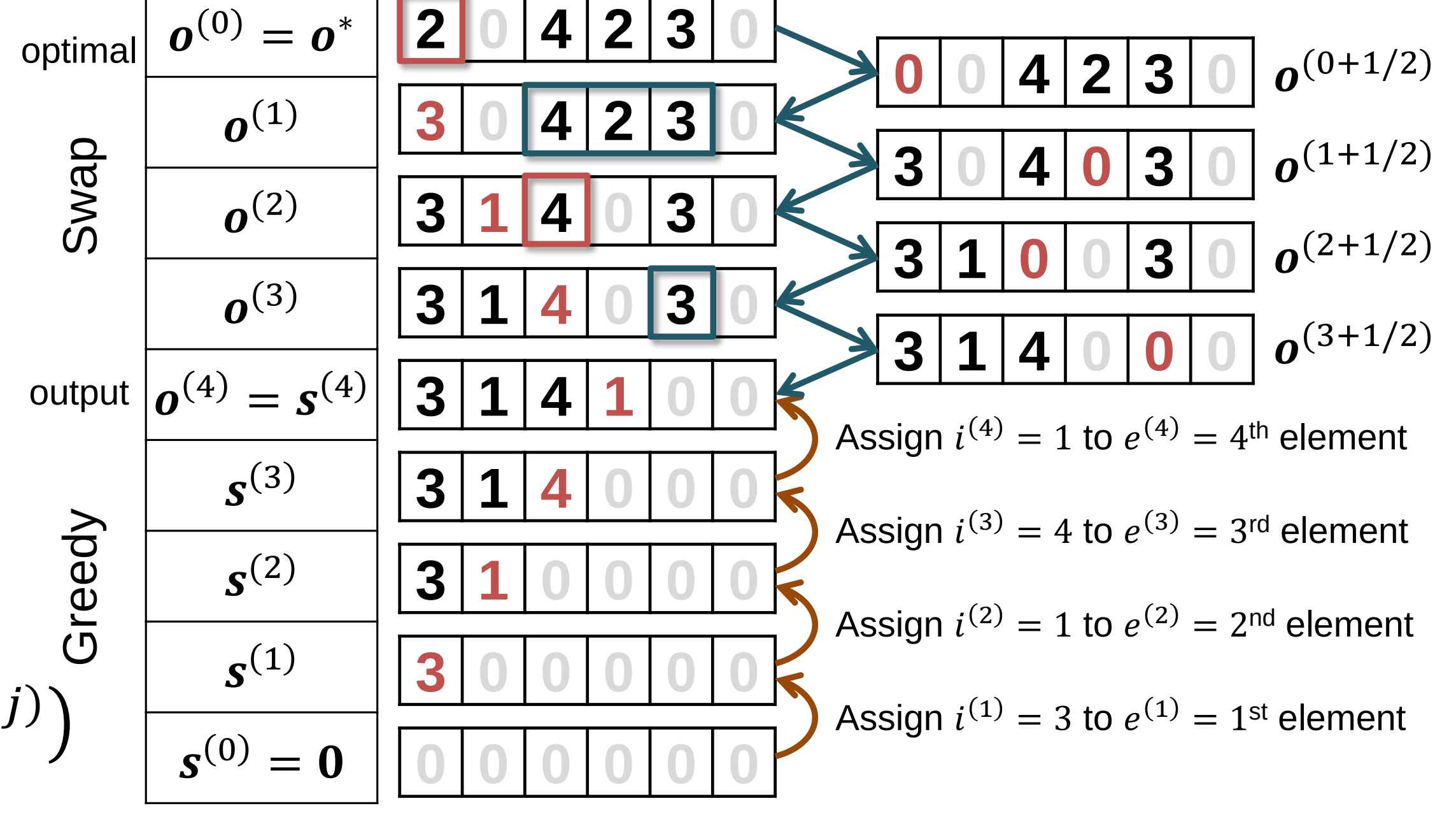
k-Greedy-TS

```
$\boldsymbol{s} \leftarrow \boldsymbol{0}$
for $j = 1$ to $B$
    $(e^{(j)}, i^{(j)}) \leftarrow \operatorname{argmax}_{e \in V \setminus \mathrm{supp}(\boldsymbol{s}), i} \Delta_{e,i} f(\boldsymbol{s})$
    $\boldsymbol{s}(e^{(j)}) \leftarrow i^{(j)}$
```

\# function eval. $= \mathcal{O}(knB) = \mathcal{O}(kn^2)$
Approx. ratio $= 1/2$ ⋯tight

$$f(\boldsymbol{s}^{(j)}) - f(\boldsymbol{s}^{(j-1)}) \geq f(\boldsymbol{o}^{(j-1)}) - f(\boldsymbol{o}^{(j)})$$

Proof sketch



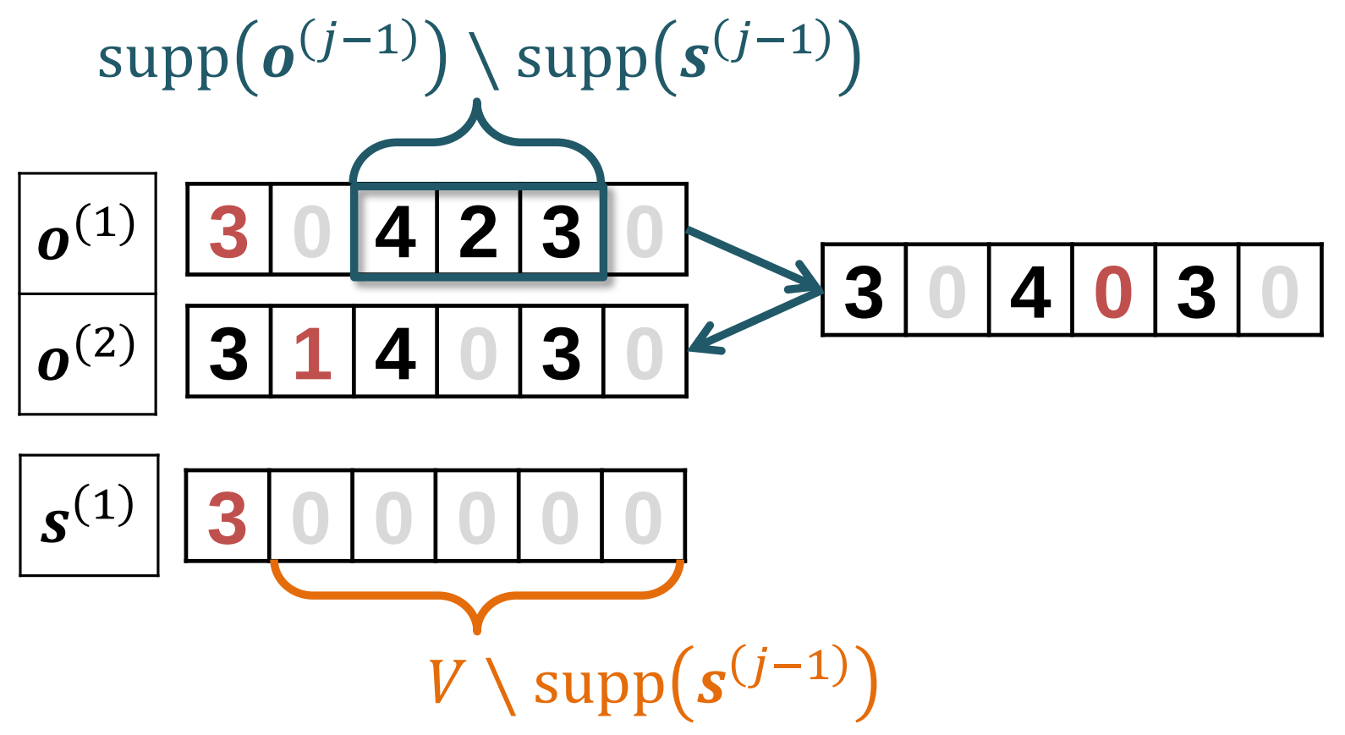
k-Stochastic-Greedy-TS

Can $O(knB)$ be reduced?

Check a small part of $V \setminus \mathrm{supp}(\boldsymbol{s})$ via **random sampling**
[Mirzasoleiman-Badanidiyuru-Karbasi-Vondrák-Krause. AAAI'15]

```
$R^{(j)} \leftarrow$ a random subset of size $\min\left\{\frac{n-j+1}{B-j+1} \log \frac{B}{\delta}, n\right\}$
        sampled from $V \setminus \mathrm{supp}(\boldsymbol{s})$
$(e^{(j)}, i^{(j)}) \leftarrow \operatorname{argmax}_{e \in R, i} \Delta_{e,i} f(\boldsymbol{s})$
```

\# function eval. $= \mathcal{O}(kn \log(B) \log(B/\delta))$
Approx. ratio $= 1/2$ w.p. $1 - \delta$



Lucky if

$$R^{(j)} \cap \left[\mathrm{supp}(\boldsymbol{o}^{(j-1)}) \setminus \mathrm{supp}(\boldsymbol{s}^{(j-1)})\right] \neq \emptyset$$
$$\left|\mathrm{supp}(\boldsymbol{o}^{(j-1)}) \setminus \mathrm{supp}(\boldsymbol{s}^{(j-1)})\right| = B - j + 1$$
$$\left|V \setminus \mathrm{supp}(\boldsymbol{s}^{(j-1)})\right| = n - j + 1$$

Individual size constraint

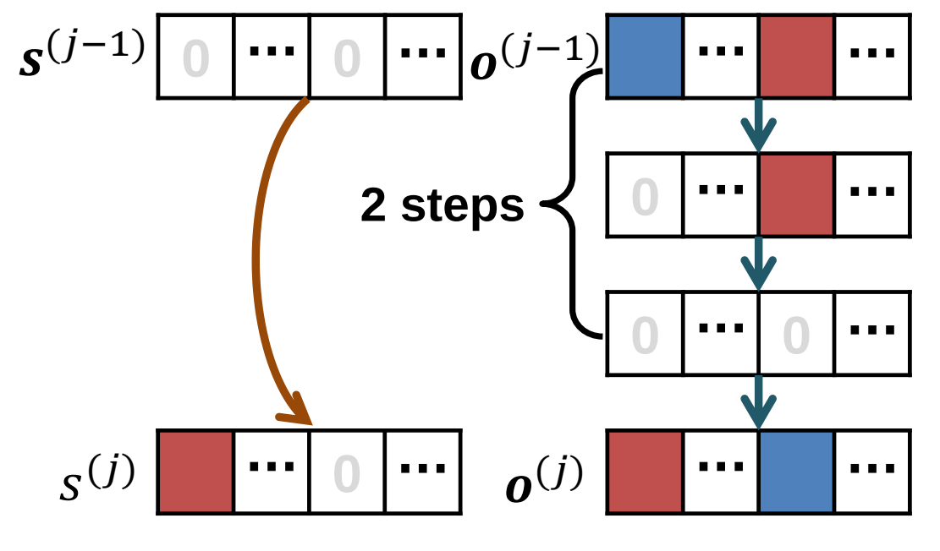
Given: a monotone k-submodular function f & k integers $B_1, \ldots, B_k$
Goal: max $f(\boldsymbol{x})$ s.t. $|\mathrm{supp}_i(\boldsymbol{x})| \leq B_i$ $(1 \leq i \leq k)$

k-Greedy-IS

```
$(e^{(j)}, i^{(j)}) \leftarrow \operatorname{argmax}_{e \in V \setminus \mathrm{supp}(\boldsymbol{s}),\ i:\ \mathrm{supp}_i(\boldsymbol{s}) < B_i} \Delta_{e,i} f(\boldsymbol{s})$
```

\# function eval. $= \mathcal{O}(knB)$
Approx. ratio $= 1/3$ ⋯tight? (open problem)

Why 1/3 ?



$$2\left(f(\boldsymbol{s}^{(j)}) - f(\boldsymbol{s}^{(j-1)})\right) \geq f(\boldsymbol{o}^{(j-1)}) - f(\boldsymbol{o}^{(j)})$$

k-Stochastic-Greedy-IS (using random sampling)

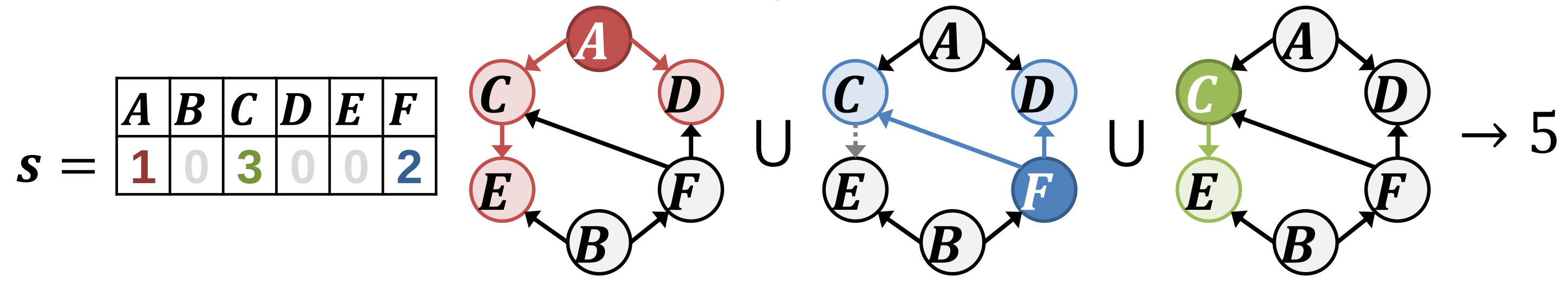
\# function eval. $= \mathcal{O}(k^2 n \log(B/k) \log(B/\delta))$
Approx. ratio $= 1/3$ w.p. $1 - \delta$

Experiment for the total size constraint

Influence maximization with k topics

Given: a social network $G = (V, E, p)$ and a budget B

How to distribute k kinds of items to B people
to maximize the spread of influence?



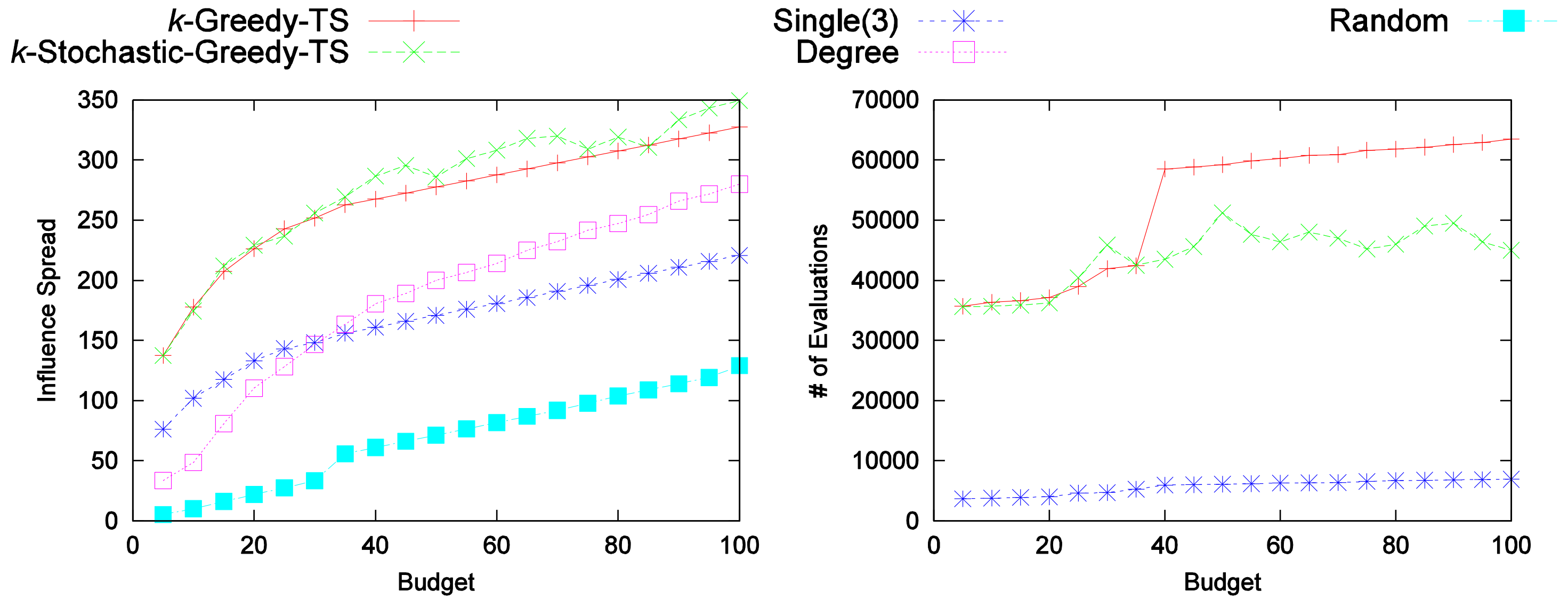
Diffusion process of the rumor on the i^{th} topic $(1 \leq i \leq k)$

0. Activate vertices in $\mathrm{supp}_i(\boldsymbol{s})$
1. An active vertex u activates an inactive vertex v w.p. $p^i_{u,v}$
2. Repeat 1

Influence spread $\sigma(\boldsymbol{s})$

Expected # vertices who eventually get active
in one of the k diffusion processes

Goal: max $\sigma(\boldsymbol{s})$ s.t. $\mathrm{supp}(S) \leq B$

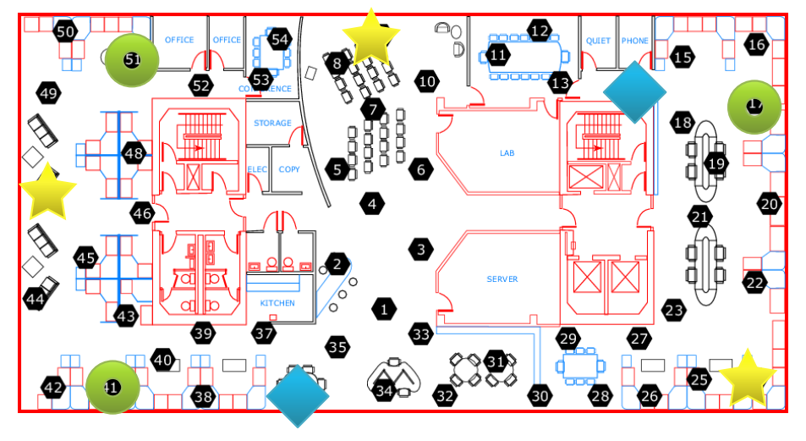


Environment: Intel Xeon E5-2690 2.90GHz CPU with 256GB memory

Experiment for the individual constraint

Sensor placement with k kinds of measures

Given:
B_1 sensors for temperature
B_2 sensors for humidity
B_3 sensors for light



How to allocate these sensors to maximize the information gain?

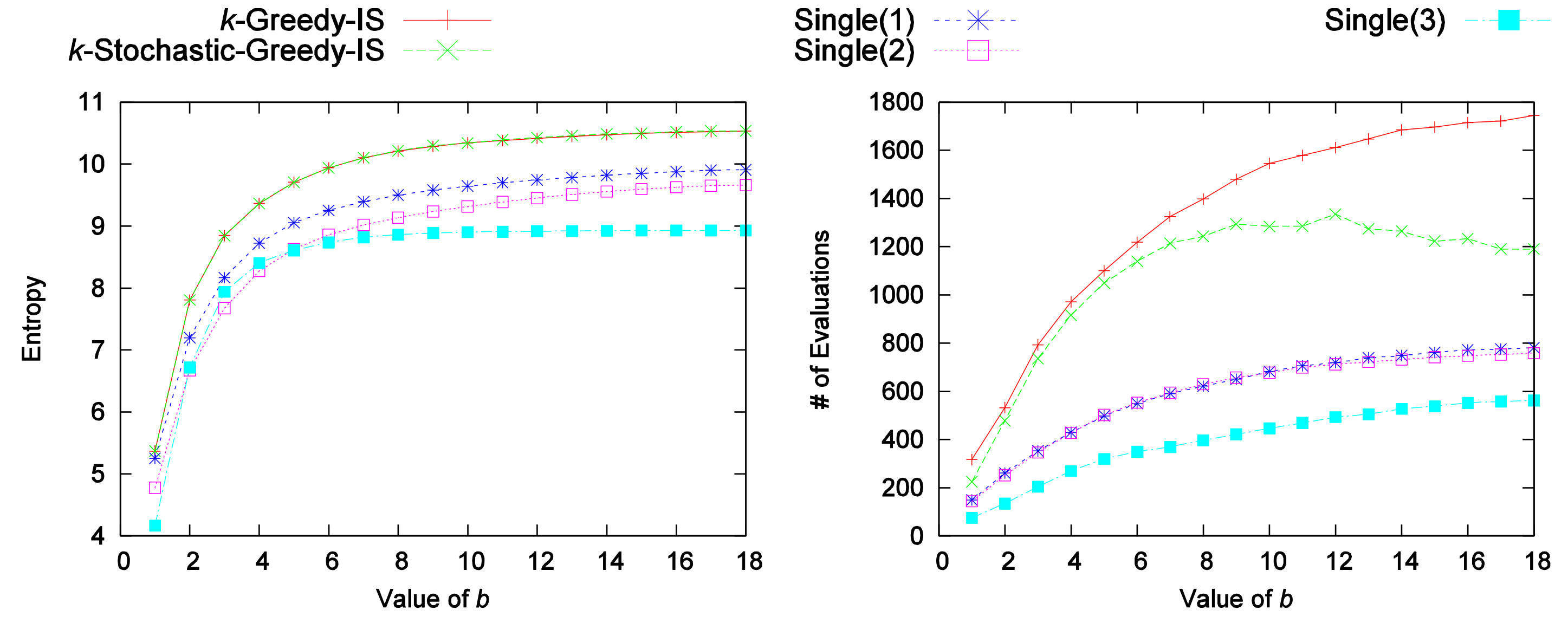
Entropy of $\mathcal{S} \subseteq \Omega = \{X_1, \ldots, X_n\}$

$$H(\mathcal{S}) = -\sum_{\boldsymbol{s} \in \mathrm{dom}\,\mathcal{S}} \Pr[\boldsymbol{s}] \log \Pr[\boldsymbol{s}]$$

$H(\Omega \mid \mathcal{S}) = H(\Omega) - H(\mathcal{S})$ measures uncertainty of Ω after observing $\mathcal{S}$

Goal: max $f(\boldsymbol{x}) = H\left(\bigcup_{e \in \mathrm{supp}(x)} \left\{X_e^{\boldsymbol{x}(e)}\right\}\right)$ s.t. $|\mathrm{supp}_i(\boldsymbol{x})| \leq B_i$

X^i_e: random variable for the observation
from a sensor of the i^{th} kind at the e^{th} location



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