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Probabilistically Checkable Reconfiguration Proofs AND Inapproximability of Reconfiguration Problems

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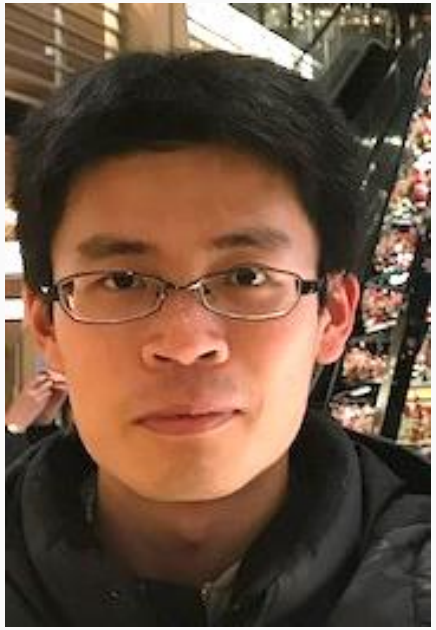
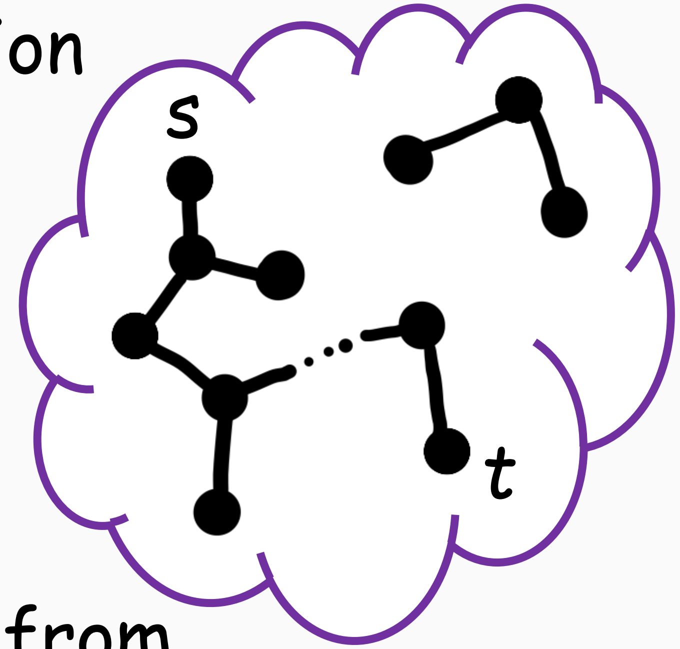


Intro of reconfiguration

Imagine **connecting** a pair of feasible solutions (of NP problem)
under a particular adjacency relation



Q. Is a pair of solutions reachable to each other?
Q. If so, what is the shortest transformation?
Q. If not, how can the feasibility be relaxed?

Many reconfiguration problems have been derived from

Satisfiability, Coloring, Vertex Cover, Clique, Dominating Set, Feedback Vertex Set, Steiner Tree, Matching, Spanning Tree, Shortest Path, Set Cover, Subset Sum, …

See [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011] [Nishimura. Algorithms 2018] [van den Heuvel. Surv. Comb. 2013] [Hoang. https://reconf.wikidot.com/]

Example 1-1

3-SAT Reconfiguration

[Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]

- **Input:** 3-CNF formula φ & satisfying σ_{ini}, σ_{tar}
- **Output:** $\boldsymbol{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
  $\sigma^{(t)}$ satisfies φ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)

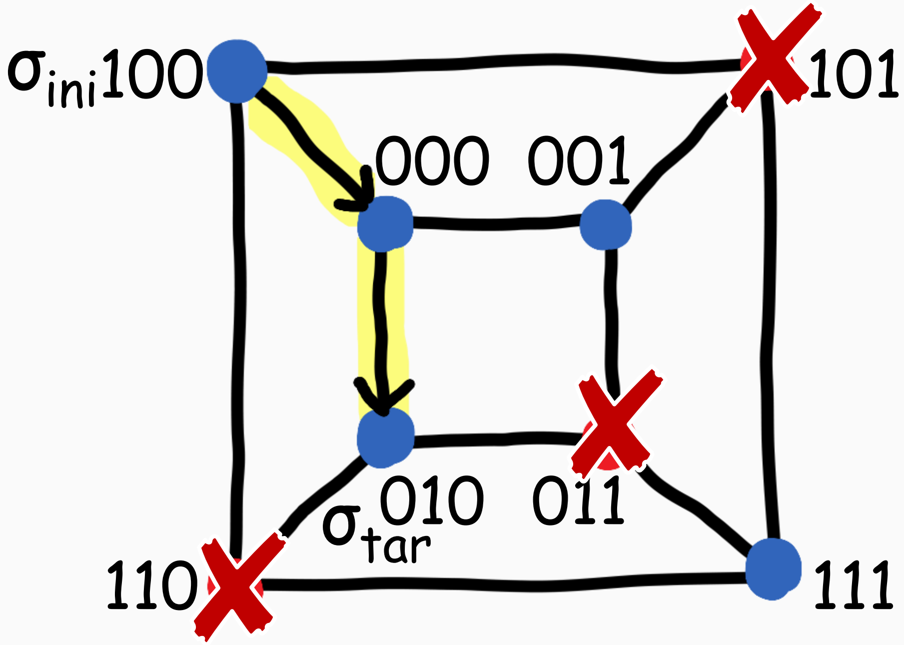
YES case

$\varphi = (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee \bar{z})$

$\sigma_{ini} = (1,0,0)$

$\sigma_{tar} = (0,1,0)$

⚠ Length of $\boldsymbol{\sigma}$ can be $2^{\Omega(\text{input size})}$



Example 1-2

3-SAT Reconfiguration

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- **Output**: $\boldsymbol{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
  $\sigma^{(t)}$ satisfies φ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)

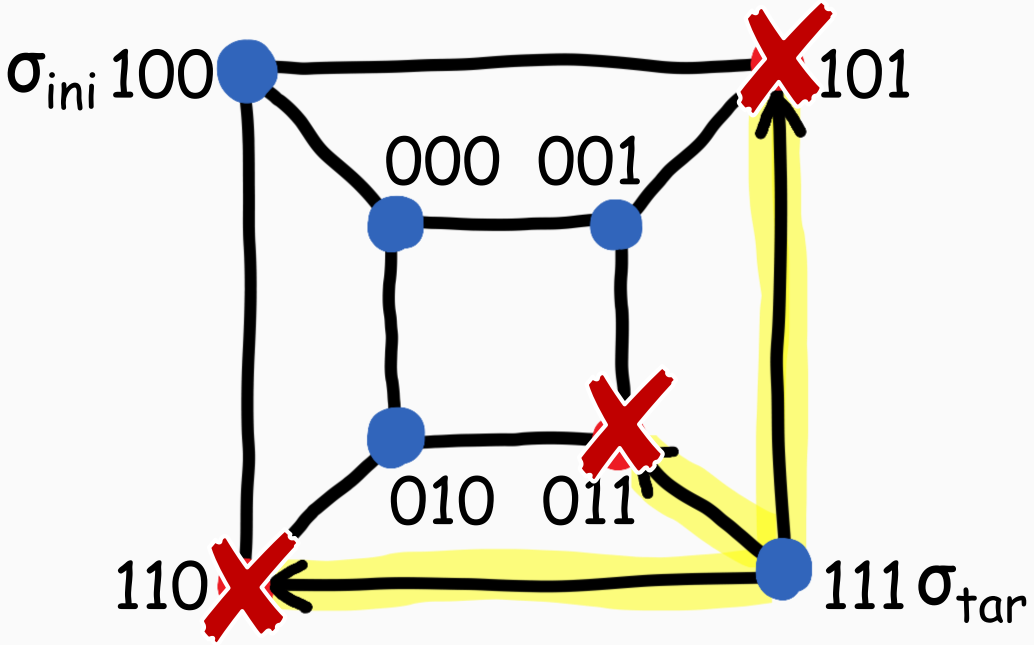
NO case

$\varphi = (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee \bar{z})$

$\sigma_{ini} = (1,0,0)$

$\sigma_{tar} = (1,1,1)$

⚠ Length of $\boldsymbol{\sigma}$ can be $2^{\Omega(\text{input size})}$

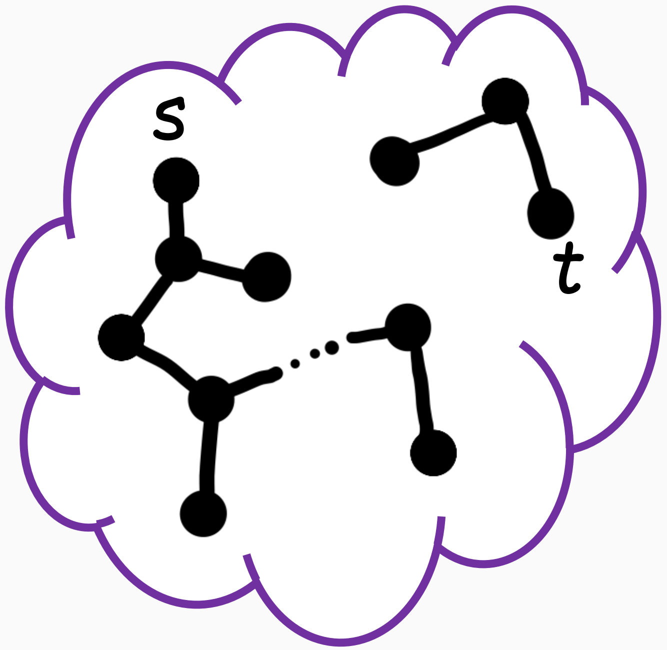


Complexity of reconfiguration problems

Source problem	Existence	Reconfiguration
Satisfiability	NP-complete	**PSPACE-complete** [Gopalan-Kolaitis-Maneva-Papadimitriou. SIAM J. Comput. 2009]
Independent Set	NP-complete	**PSPACE-complete** [Hearn-Demaine. Theor. Comput. Sci. 2005]
Matching	P	**P** [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]
3-Coloring	NP-complete	**P** [Cereceda-van den Heuvel-Johnson. J. Graph Theory 2011]
Shortest Path	P	**PSPACE-complete** [Bonsma. Theor. Comput. Sci. 2013]
Independent Set on bipartite graphs	P	**NP-complete** [Lokshtanov-Mouawad. ACM Trans. Algorithms 2019; SODA 2018]

👆 **Nontrivial relation**

Optimization versions of reconfiguration problems



Even if…

- 😣 NOT reconfigurable! and/or
- 😣 many problems are **PSPACE-complete**!

Still want an "approximate" reconf. sequence

(e.g.) made up of almost-satisfying assignments

↓

RELAX feasibility to obtain approximate reconfigurability

e.g. Set Cover Reconf. [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Subset Sum Reconf. [Ito-Demaine. J. Comb. Optim. 2014]

Submodular Reconf. [O.-Matsuoka. WSDM 2022]

Example 1+

Maxmin 3-SAT Reconfiguration

[Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

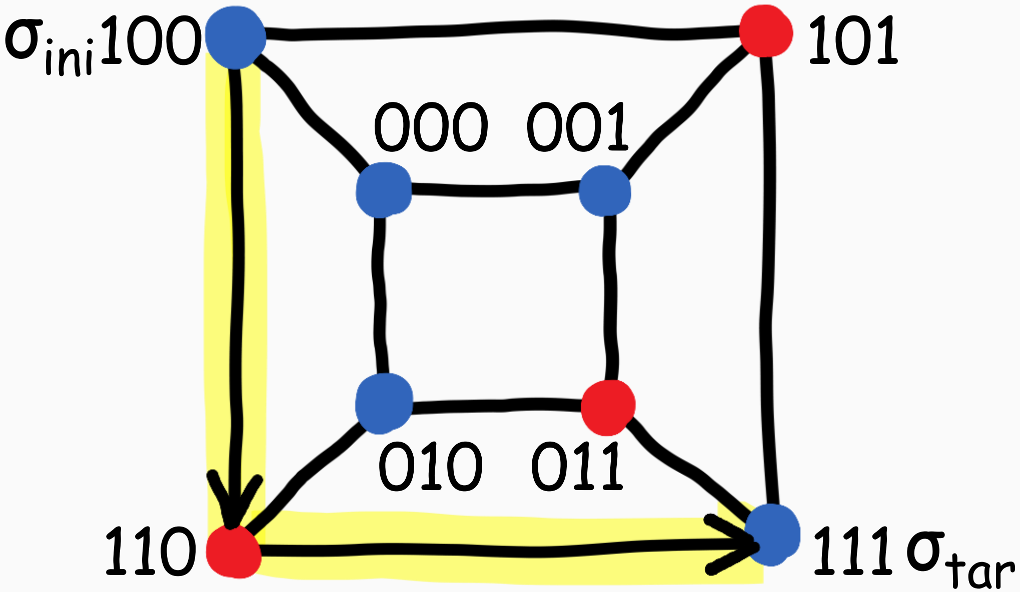
- **Input:** 3-CNF formula φ & satisfying σ_{ini}, σ_{tar}
- **Output:** $\boldsymbol{\sigma} = (\sigma^{(1)}=\sigma_{ini}, \ldots, \sigma^{(T)}=\sigma_{tar})$ (reconf. sequence) s.t.
  ~~$\sigma^{(t)}$ satisfies φ~~ (feasibility)
  $\mathrm{Ham}(\sigma^{(t)}, \sigma^{(t+1)}) = 1$ (adjacency on hypercube)
- **Goal:** $\max_{\boldsymbol{\sigma}} \mathrm{val}_\varphi(\boldsymbol{\sigma}) := \min_t$ (frac. of satisfied clauses by $\sigma^{(t)}$)

$\varphi = (\bar{x} \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee \bar{z})$

- $\sigma_{ini} = (1,0,0)$
- $\sigma_{tar} = (1,1,1)$

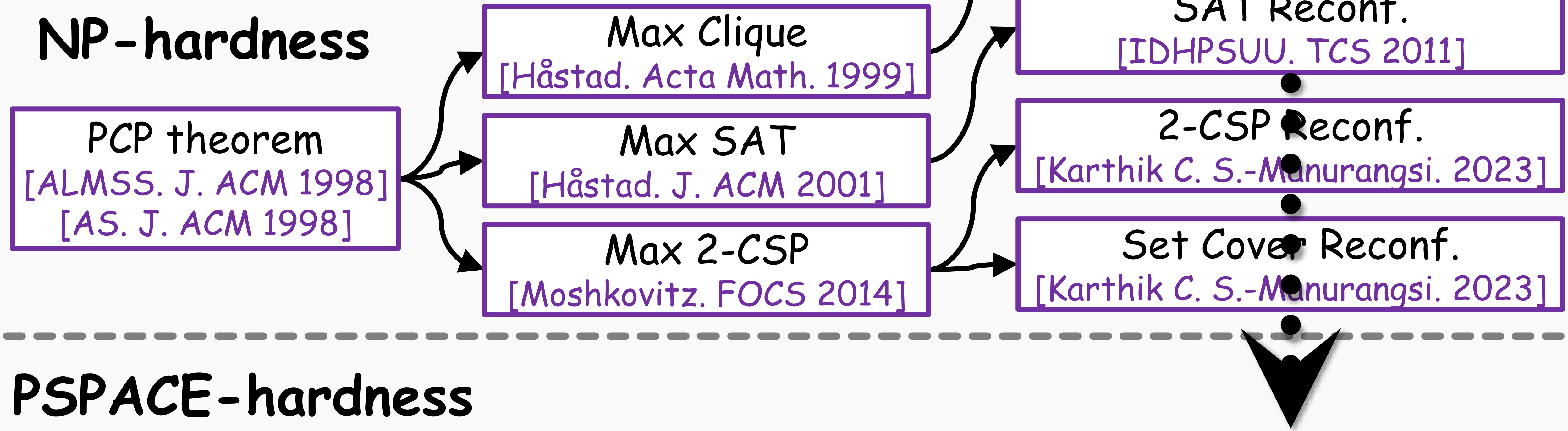
➜ $\mathrm{val}_\varphi(\boldsymbol{\sigma}) = \min\{1, \frac{2}{3}, 1\} = \frac{2}{3}$

⚠ Length of $\boldsymbol{\sigma}$ can be $2^{\Omega(\text{input size})}$

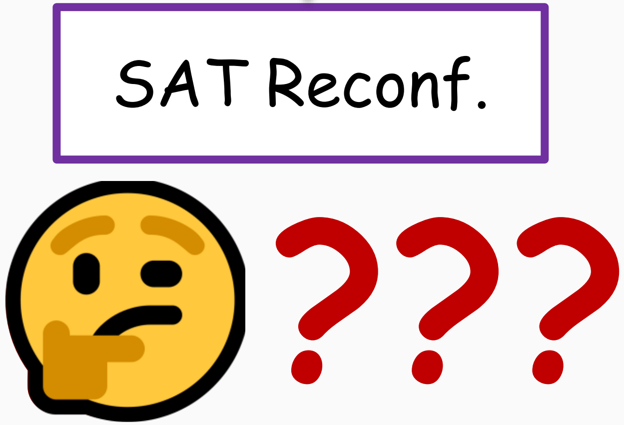


Known results on hardness of **approximation**

NP-hardness



PSPACE-hardness

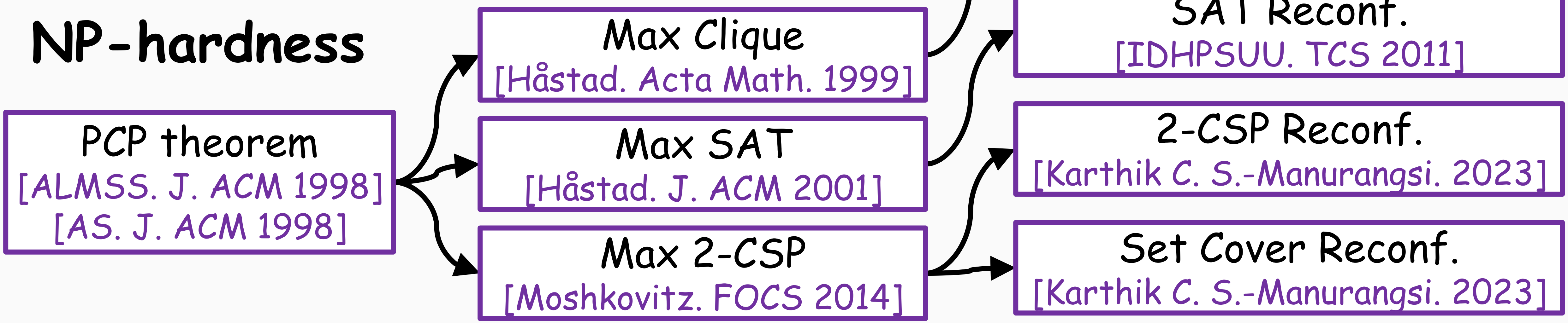


Why do we need **PSPACE**-hardness?

- no polynomial-time algorithm (**P ≠ PSPACE**)
- no polynomial-length sequence (**NP ≠ PSPACE**)

Known results on hardness of approximation

NP-hardness



5. Open problems [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. ISAAC 2008 & TCS 2011]

There are many open problems raised by this work, and we mention some of these below:

- Can the MATCHING RECONFIGURATION problem for edge-weighted graphs be solved also in polynomial time? We conjecture that the answer is positive.
- Is the TRAVELING SALESMAN RECONFIGURATION problem (where two tours are adjacent if they differ in two edges) PSPACE-complete?
- Are there better approximation algorithms for the MINMAX POWER SUPPLY RECONFIGURATION problem? Lower bounds?
- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

Known results on hardness of **approximation**

NP-hardness

PCP theorem
[ALMSS. J. ACM 1998]
[AS. J. ACM 1998]

Max Clique
[Håstad. Acta Math. 1999]

Max SAT
[Håstad. J. ACM 2001]

Max 2-CSP
[Moshkovitz. FOCS 2014]

Clique Reconf.
[IDHPSUU. TCS 2011]

SAT Reconf.
[IDHPSUU. TCS 2011]

2-CSP Reconf.
[Karthik C. S.-Manurangsi. 2023]

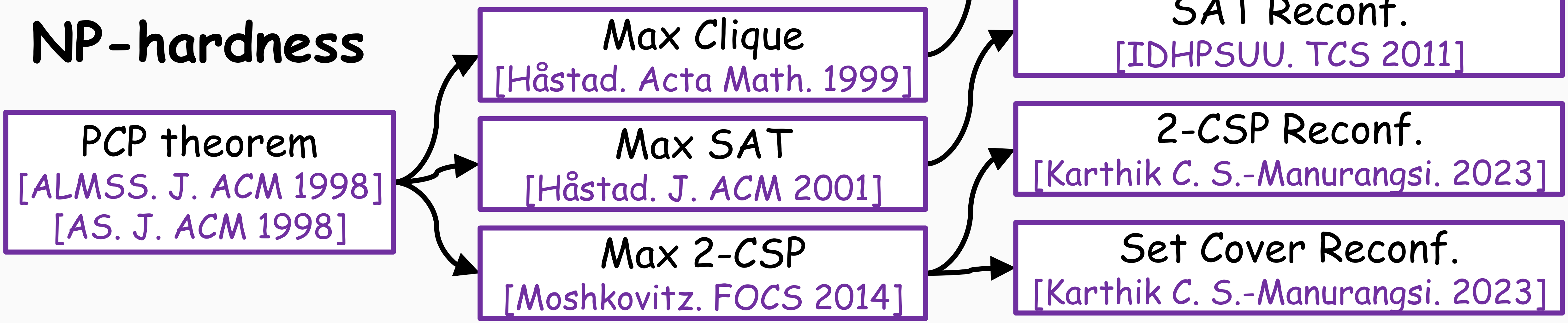
Set Cover Reconf.
[Karthik C. S.-Manurangsi. 2023]

PSPACE-hardness

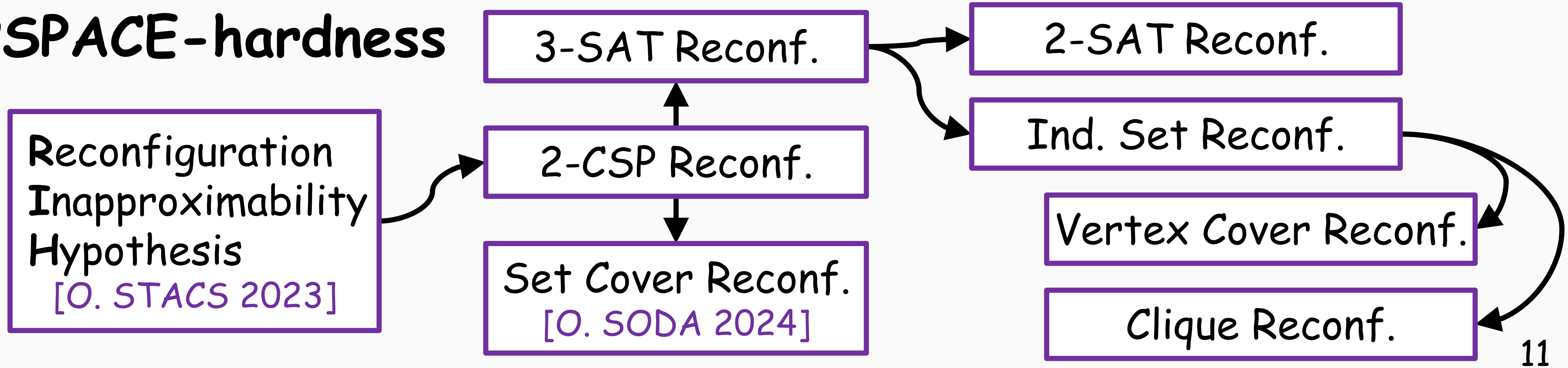
🎯Reconf. analogue of PCP theorem

Known results on hardness of **approximation**

NP-hardness



PSPACE-hardness



Known results on hardness of **approximation**

NP-hardness

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SAT Reconf.
[IDHPSUU. TCS 2011]

Reconf.
-Manurangsi. 2023]

Reconf.
urangsi. 2023]

Is RIH true?

PSPACE

-SAT Reconf.

Reconfiguration Inapproximability Hypothesis
[O. STACS 2023]

2-CSP Reconf.

Set Cover Reconf.
[O. SODA 2024]

Ind. Set Reconf.

Vertex Cover Reconf.

Clique Reconf.

Our results

In a nutshell:

😋 Reconfiguration Inapproximability Hypothesis is true

➜ Resolve 4th open problem of [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

⚠️Independent of [Karthik C. S.-Manurangsi. 2023]

Technically…

Probabilistically Checkable Reconfiguration Proofs

A new PCP-like characterization of **PSPACE**

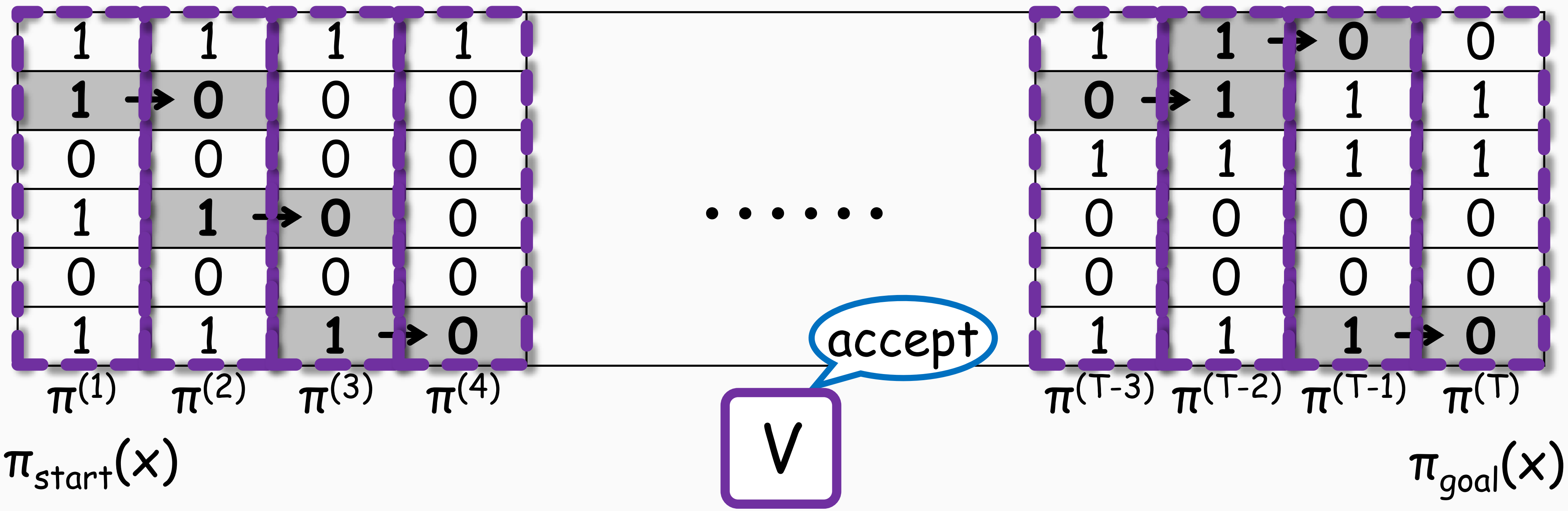
Probabilistically Checkable Reconfiguration Proofs… What's that?

- Verifier V & poly-time alg. π_{start} & π_{goal} for language $L \subseteq \{0,1\}^*$

(Completeness)

$x \in L \implies \exists\, \boldsymbol{\pi} = (\pi^{(1)}, \dots, \pi^{(T)})$ from $\pi_{start}(x)$ to $\pi_{goal}(x)$ s.t.

$$\forall t \;\; \Pr[V(x) \text{ accepts } \pi^{(t)}] = 1$$



Probabilistically Checkable Reconfiguration Proofs... What's that?

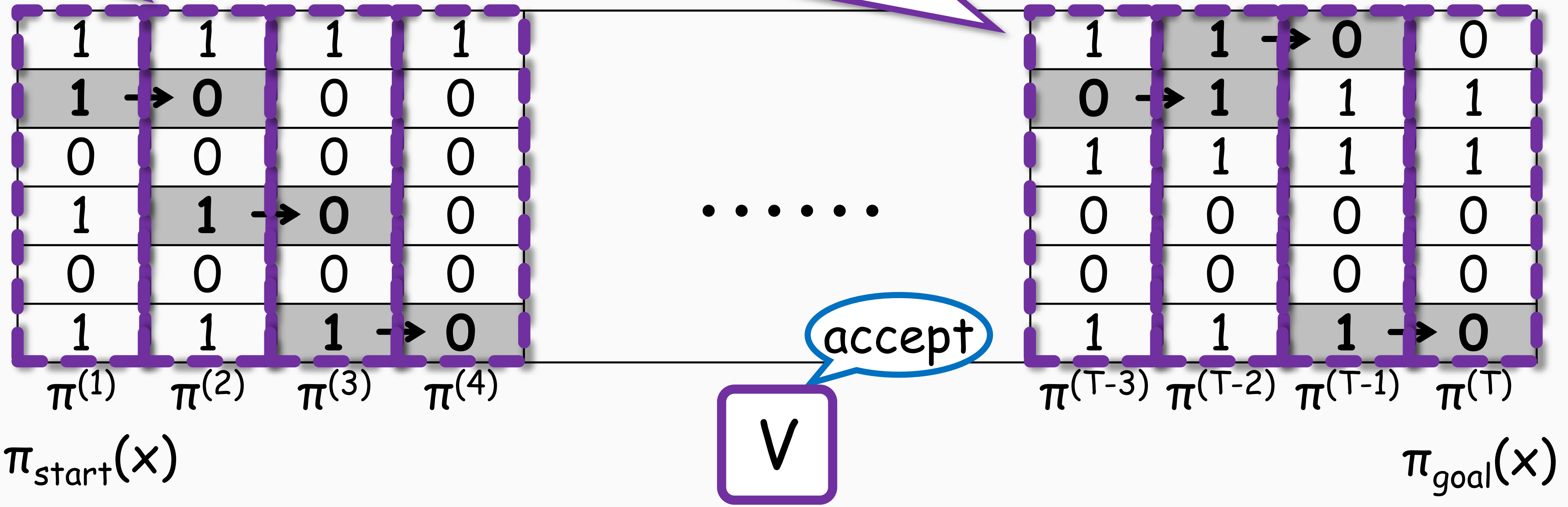
- Verifier V & poly-time alg. π_{start} & π_{goal} for language $L \subseteq \{0,1\}^*$

Adjacent proofs differ in (at most) one symbol

$\pi^{(T)}$) from ...

Pr[V...

π can be exponentially long



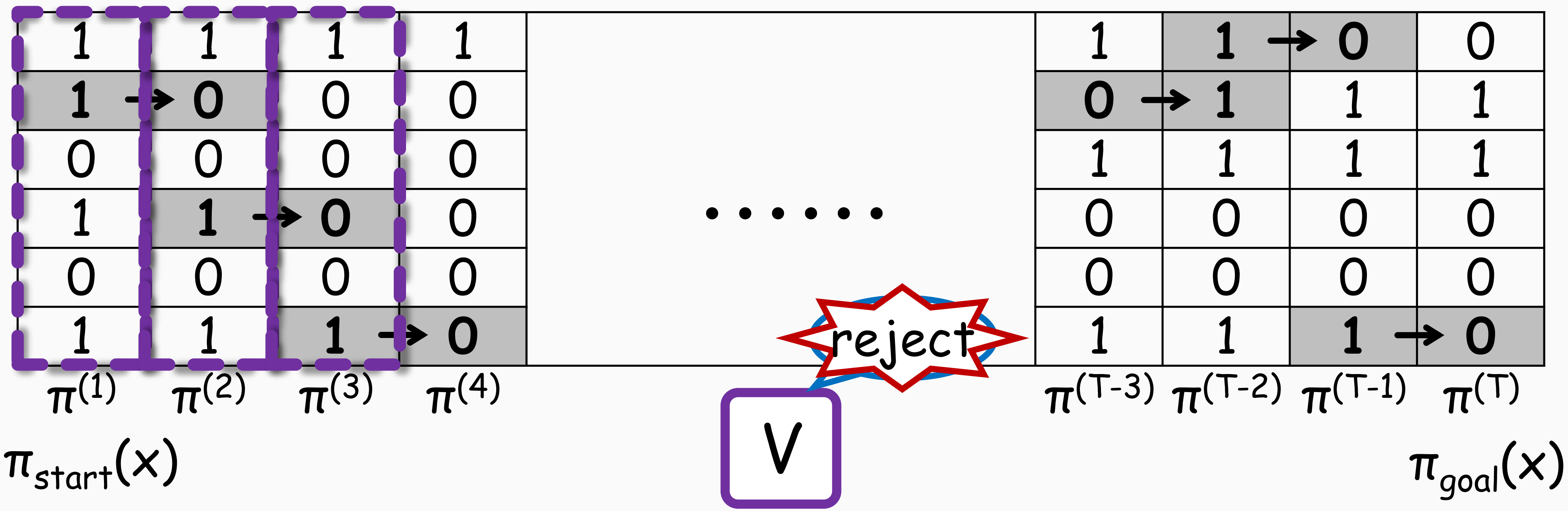
Probabilistically Checkable Reconfiguration Proofs... What's that?

- Verifier V & poly-time alg. π_{start} & π_{goal} for language $L \subseteq \{0,1\}^*$

(Soundness)

$x \notin L \implies \forall \boldsymbol{\pi} = (\pi^{(1)}, \dots, \pi^{(T)})$ from $\pi_{start}(x)$ to $\pi_{goal}(x)$,

$$\exists t \ \Pr[V(x) \text{ accepts } \pi^{(t)}] < \tfrac{1}{2}$$



PCRP Theorem

$$\mathbf{PSPACE} = \mathbf{PCRP}[O(\log n), O(1)]$$

$$L \in \mathbf{PSPACE}$$

$$\Updownarrow$$

$\exists$ Verifier V with randomness comp. $O(\log n)$ & query comp. $O(1)$

$\exists$ poly-time alg. π_{start} & π_{goal}

completeness $= 1$ & soundness $< \frac{1}{2}$ for L

🤪Quick Q&A

Q. Any intuition/interpretation?

A. $\forall\cdot$coRP-type verifier: Guess t & check $\pi^{(t)}$ probabilistically

cf. PCP for **NEXP** ($\supseteq$ **PSPACE**) [Babai-Fortnow-Lund. Comput. Complex. 1991]

random bits = $n^{\Theta(1)}$

cf. PCP for **NP** ($\subseteq$ **PSPACE**) [ALMSS. J. ACM 1998] [AS. J. ACM 1998]

Not using nondeterminism ($\forall$)

"$\forall$" of PCRP cannot be replaced by random choice (see our paper)

Q. Can Dinur's gap amplification [Dinur. J. ACM 2007] be adapted?

A. Partial progress made [O. STACS 2023 & SODA 2024] but still fails…

Proof sketch

Starting point: Succinct Graph Reachability

- **Input:** Graph $G=(V,E)$ over $\{0,1\}^n$ & vertices $v_{start}, v_{goal} \in \{0,1\}^n$
- **Output:** $(x^{(1)} \circ y^{(1)} = v_{start} \circ v_{start}, \ldots, x^{(T)} \circ y^{(T)} = v_{goal} \circ v_{goal})$ (reconf. sequence)

  s.t. $(x^{(t)}, y^{(t)}) \in E$ OR $x^{(t)} = y^{(t)}$ (feasibility)

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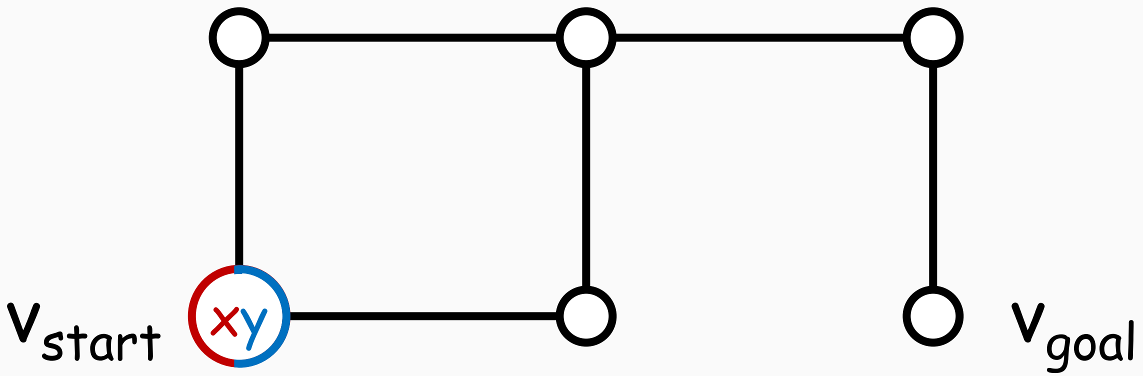

⚠ Succinct Graph Reachability is **PSPACE**-complete

[Galperin-Wigderson. Inf. Control. 1983] [Papadimitriou-Yannakakis. Inf. Control. 1986]

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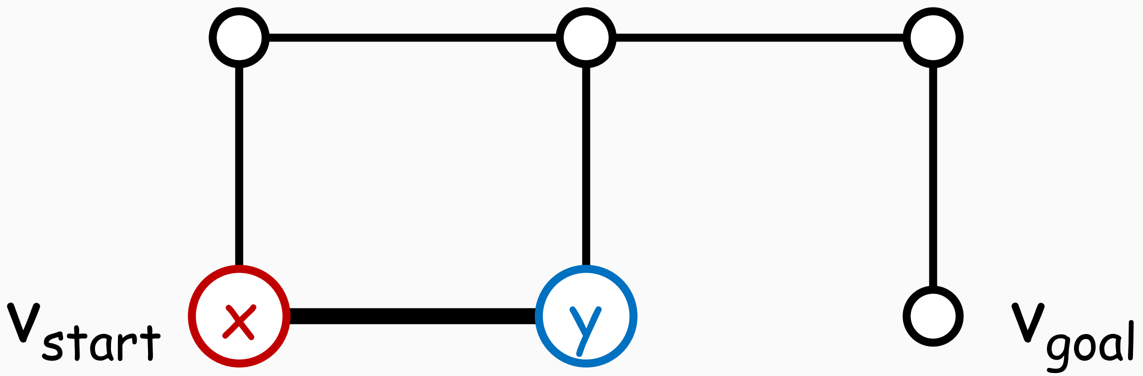

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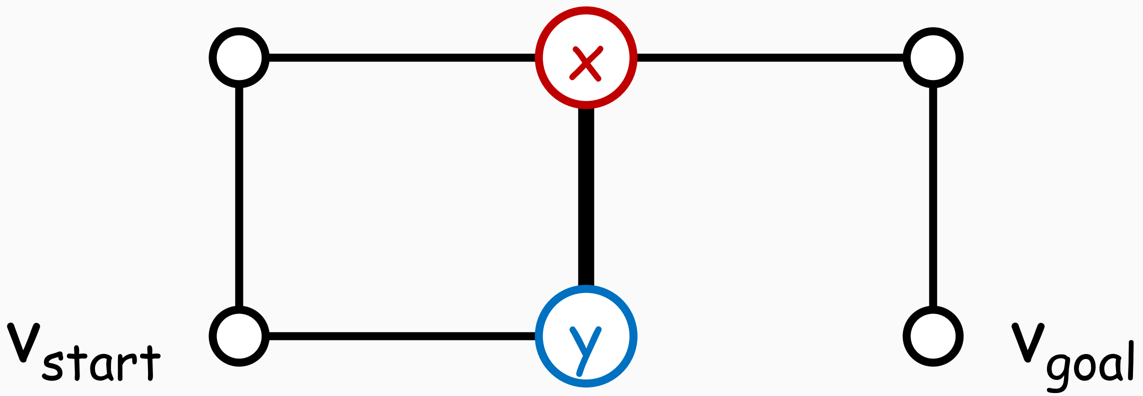

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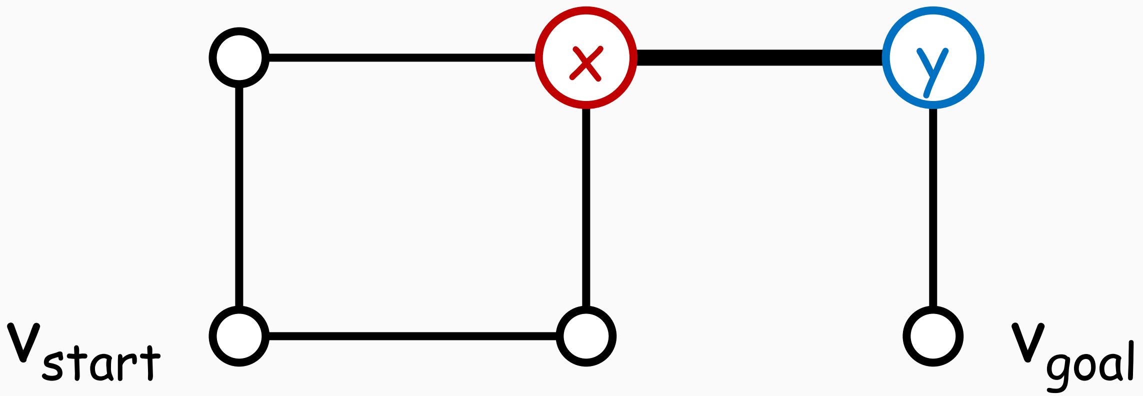

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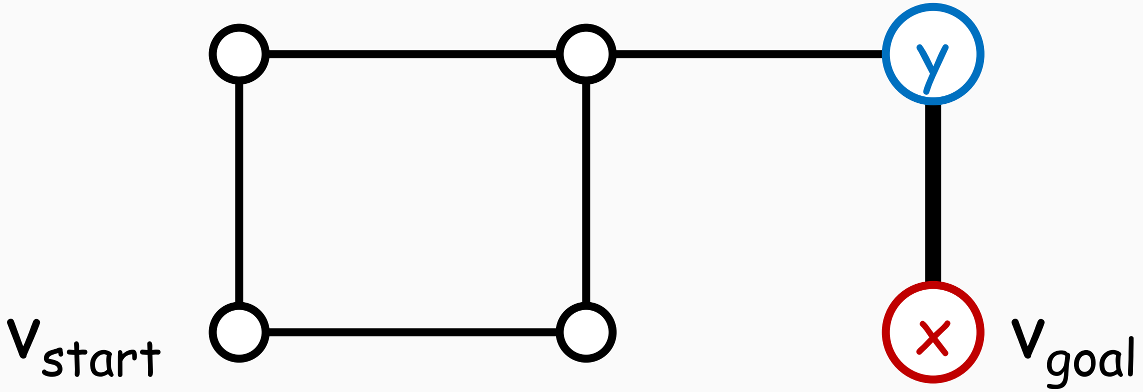

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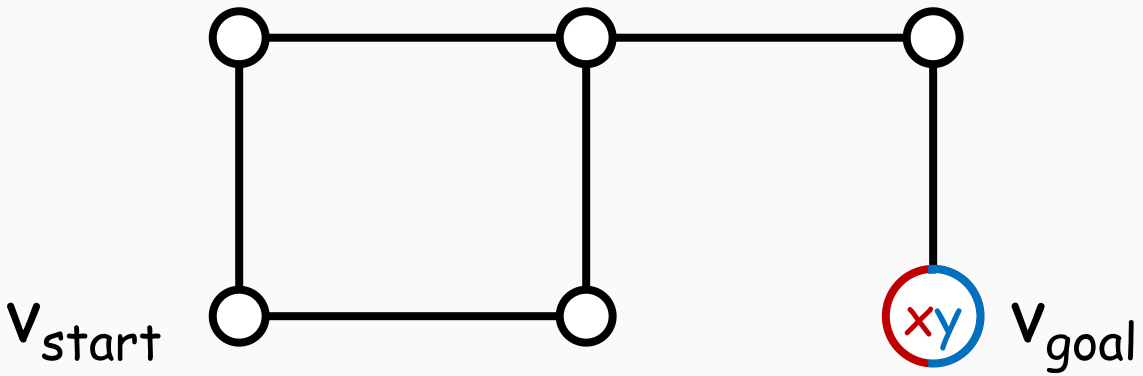

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Proof sketch

To construct PCRP for Succinct Graph Reachability…

- 🎯 "$(x^{(t)}, y^{(t)}) \in E$" should be probabilistically checkable
- 😋 Encode by **error-correcting code** Enc: $\{0,1\}^n \to \{0,1\}^\ell$ & use **PCP of proximity** (a.k.a. assignment testers)
  [Ben-Sasson; Goldreich; Harsha; Sudan; Vadhan. SIAM J. Comput. 2006]
  [Dinur-Reingold. SIAM J. Comput. 2006]
  PCPP for $L_G := \{\text{Enc}(x) \circ \text{Enc}(y) : (x, y) \in E \text{ OR } x = y\}$

- 🎯 Any adjacent pair of proofs differs in (at most) one symbol
- 😋 Introduce "**in transition**" symbol $\perp \neq 0,1$

Proof sketch

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- 🎯 "$(x^{(t)}, y^{(t)}) \in E$" should be probabilistically checkable
- 😋 Encode by error-correcting code Enc: $\{0,1\}^n \to \{0,1\}^\ell$ &

$\mathrm{Enc}(x) \circ \mathrm{Enc}(\mathbf{y}) \circ \boldsymbol{\pi}_{xy}$	PCPP accepts w.p. 1
$\mathrm{Enc}(x) \circ ????? \circ ??$ 1%-far from Enc	PCPP rejects w.p. 1%
$\mathrm{Enc}(x) \circ \mathrm{Enc}(\mathbf{z}) \circ \boldsymbol{\pi}_{xz}$	PCPP accepts w.p. 1

- 🎯 Any adjacent pair of proofs differs in (at most) one symbol
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Proof sketch

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😋 Encode by error-correcting code Enc: $\{0,1\}^n \to \{0,1\}^\ell$ &

$\mathrm{Enc}(x) \circ \mathrm{Enc}(\mathbf{y}) \circ \boldsymbol{\pi}_{xy}$	PCPP accepts w.p. 1
$\mathrm{Enc}(x) \circ \bot\bot\bot\bot \circ \pi_{xy}$	If we see $\bot$, do NOT run PCPP
	😋 π can be arbitrarily changed
$\mathrm{Enc}(x) \circ \bot\bot\bot\bot \circ \pi_{xz}$	If we see $\bot$, do NOT run PCPP
$\mathrm{Enc}(x) \circ \mathrm{Enc}(\mathbf{z}) \circ \boldsymbol{\pi}_{xz}$	PCPP accepts w.p. 1

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Conclusions

Probabilistically Checkable **Reconfiguration** Proofs

- A new PCP-like characterization of **PSPACE**

- **PSPACE**-hardness of approximating reconfiguration problems
  Resolve **RIH** [O. STACS 2023] & 4^{th} open problem of
  [Ito-Demaine-Harvey-Papadimitriou-Sideri-Uehara-Uno. Theor. Comput. Sci. 2011]

Other applications?

- Pebble games [Paterson-Hewitt. 1970]
  Proof complexity [Nordström. Log. Methods Comput. Sci. 2013]
  PSPACE-hardness of additive approx. is known
  [Chan-Lauria-Nordströmm-Vinyals. FOCS 2015] [Demaine-Liu. WADS 2017]